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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, E. C. THOMPSON

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ON H-SPACES AND THEIR HOMOTOPY GROUPS

By I. M. JAMES (*Oxford*)

[Received 3 February 1959; revised† 1 December 1959]

1. Introduction

LET A be a space with basepoint‡ a_0, and let $m: A \times A \to A$ be a map. If

$$m(x, a_0) = x = m(a_0, x) \quad (x \in A),$$

we describe m as a *multiplication* and the pair (A, m) as an *H-space*. Except when comparing various multiplications on the same space we generally write $x.y$ instead of $m(x, y)$ and refer to A itself as an H-space. Two multiplications are said to be *equivalent* if they are homotopic regarded as maps of $A \times A$ into A. The *inversion* of m is the multiplication m' such that $m(x, y) = m'(y, x)$. We say that m is *homotopy-commutative* if m is equivalent to m'. We say that m is *homotopy-associative* if

$$f \simeq g: A \times A \times A \to A,$$

where f, g are the maps given by

$$f(x, y, z) = m(m(x, y), z), \qquad g(x, y, z) = m(x, m(y, z)).$$

Notice that m' is homotopy-associative if m is. Multiplications can be lifted to covering spaces, in the obvious way, and the result is homotopy-commutative or homotopy-associative according as the original is. If B is an H-space and A is the space of loops on B, then A inherits from B a multiplication which is both homotopy-commutative and homotopy-associative.

Let $u, v: X \to A$ be maps, where X is a space and A is an H-space. We define the product map $w: X \to A$ by $w(x) = u(x).v(x)$, using the multiplication in A, and write $w = u.v$. Notice that the constant map acts as two-sided identity. The homotopy classes of maps of X into A form a set $\pi(X; A)$, which inherits a binary operation from the multiplication of maps. The class of nul-homotopic maps acts as identity. The operation is commutative or associative according as the multiplication on A is homotopy-commutative or homotopy-associative.

† The revision was made after some stimulating conversation with Dr. M. Sugawara and Professor G. W. Whitehead, to whom my thanks are due.

‡ We work in the category of spaces with basepoints. Every subspace must contain the basepoint, and all maps and homotopies must respect basepoints. The basepoint in a product space is the product of the basepoints.

Recall that a *quasi-group* is a set M with a binary operation, written multiplicatively, such that the equations

$$x.a = b, \qquad a.y = b \quad (a, b \in M)$$

admit one and only one pair of solutions x, y of M. *A loop* is a quasi-group with a two-sided identity. We shall prove the theorem:

THEOREM 1.1. *Let K be a CW-complex and let A be an H-space. Then $\pi(K; A)$ forms a loop with multiplication inherited from A and identity the class of nul-homotopic maps.*

Since an associative loop is a group, we deduce at once the corollary:

COROLLARY 1.2. *If the multiplication on A is homotopy-associative, then $\pi(K; A)$ forms a group.*

Consequently the theory of group-like spaces, as expounded in (**15**), can be carried through for homotopy-associative H-spaces as well; there is no need to assume that any homotopy-inverses exist. Thus relations can be established, for example, between the Lusternik–Schnirelmann category of K and the nilpotency class of $\pi(K; A)$. Without assuming the multiplication to be homotopy-associative it is still possible to carry through some of the arguments of (**15**), with modifications, but it is difficult to express the results neatly in terms of loop-theory.

A loop admits left and right inverses both of which are unique. By taking $K = A$ in Theorem 1.1 we deduce the corollary [cf. § 4 of (**10**)]:

COROLLARY 1.3. *If A is a CW-complex, then the multiplication on A admits left and right homotopy-inverses, and these are unique up to homotopy. Moreover, the two homotopy classes coincide when the multiplication is homotopy-associative.*

The proof of Theorem 1.1 is given in § 4 below, where it appears as an application of more general results on the homotopy classification of maps into H-spaces. In the rest of this note we study the case when K is a product of spheres so as to get some idea of the structure involved. In § 6, following (**9**) and (**15**), we define a homomorphism

$$\pi_p(A) \otimes \pi_q(A) \to \pi_{p+q}(A),$$

which is trivial if the multiplication is homotopy-commutative. There is a law of commutation and, whenever the multiplication is homotopy-associative, there is a form of the Jacobi identity. The corresponding results for group-like spaces are well known from (**15**). In § 7 we define a homomorphism

$$\pi_p(A) \otimes \pi_q(A) \otimes \pi_r(A) \to \pi_{p+q+r}(A),$$

which is trivial if the multiplication is homotopy-associative. Certain permutation laws hold when the multiplication is homotopy-commutative. Combinations of these two invariants are studied in § 8, and a relation is established between them which reduces to the Jacobi identity when the multiplication is homotopy-associative. In § 9 the theory is illustrated by considering examples when A is a sphere. There is an appendix containing some supplementary theorems on the structure of $\pi(X; A)$ in case X or A is a suspension.

2. Separation elements

Let V^{n+1} $(n \geqslant 0)$ denote the $(n+1)$-element which consists of points $(x_0, x_1, \dots, x_i, \dots)$ in Hilbert space such that $x_i = 0$ if $i > n$ and such that

$$x_0^2 + x_1^2 + \dots + x_n^2 \leqslant 1.$$

We take S^n, the standard n-sphere, to be the boundary of V^{n+1}. Let e denote the point of S^n where $x_0 = -1$, and let E^n_+, E^n_- denote the hemispheres where $x_n \geqslant 0$, $x_n \leqslant 0$, respectively. Let $p_n, q_n: V^n \to S^n$ $(n \geqslant 1)$ denote projections parallel to the axis of x_n onto E^n_+, E^n_-, respectively. We adopt the system of orientations described in (**4**), where the degrees of these projections are $(-1)^n$, $(-1)^{n+1}$, respectively [see (1.2) of (**4**)].

Separation elements are defined as follows. Let K be a CW-complex† with a subcomplex L whose complement is an oriented n-cell e^n. Thus e^n is the homeomorphic image of the interior of V^n under a characteristic map $f: V^n \to K$ such that $fS^{n-1} \subset L$. Let X be a space and let $u, v: K \to X$ be maps which agree on L. The separation element $d(u, v)$ of $\pi_n(X)$ is defined to be the class of the map $g: S^n \to X$ which is given by

$$gp_n = uf, \qquad gq_n = vf. \tag{2.1}$$

For $d(u, v) = 0$ it is necessary and sufficient that $u \simeq v$ (rel L). The main properties of separation elements are the addition formula and various forms of naturality, as set out in § 10 of (**5**).

Let $p'_m, q'_m: V^m \times K \to S^m \times K$ denote the products of p_m, q_m, respectively, with the identity on K, so that

$$p'_m(V^m \times K) = E^m_+ \times K, \qquad q'_m(V^m \times K) = E^m_- \times K.$$

Then p'_m determines a relative homeomorphism of

$$(V^m \times K,\ S^{m-1} \times K \cup V^m \times L) \quad \text{into} \quad (S^m \times K,\ E^m_- \times K \cup S^m \times L),$$

which contains $(S^m \times K,\ e \times K \cup S^m \times L)$ as a deformation retract. The degree of p'_m being $(-1)^m$ we deduce the lemma:

† In a complex the basepoint is required to be a sub-complex.

LEMMA 2.2. *Let $u, v: S^m \times K \to X$ be maps which agree on*

$$E^m_- \times K \cup S^m \times L.$$

Then
$$d(u, v) = (-1)^m d(up'_m, vp'_m).$$

In this lemma the elements $d(u, v)$, $d(up'_m, vp'_m)$ are evaluated with respect to the subspace $e \times K \cup S^m \times L$, $S^{m-1} \times K \cup V^m \times L$, respectively. Similarly we obtain the lemma:

LEMMA 2.3. *Let $u, v: S^m \times K \to X$ be maps which agree on*

$$E^m_+ \times K \cup S^m \times L.$$

Then
$$d(u, v) = (-1)^{m+1} d(uq'_m, vq'_m).$$

These lead to the theorem:

THEOREM 2.4. *Let $u_i, v_i: S^m \times K \to X$ $(i = 0, 1, 2)$ be maps such that*

$$u_i | S^m \times L = v_i | S^m \times L.$$

On $E^m_+ \times K$ suppose that

$$u_0 = u_1, \quad v_0 = v_1, \quad u_2 = v_2;$$

and on $E^m_- \times K$ suppose that

$$u_0 = u_2, \quad v_0 = v_2, \quad u_1 = v_1.$$

Then
$$d(u_0, v_0) = d(u_1, v_1) + d(u_2, v_2).$$

For let $w: S^m \times K \to X$ map $E^m_+ \times K$ according to u_0 and $E^m_- \times K$ according to v_0. Then

$$\begin{aligned} d(w, v_0) &= (-1)^m d(wp'_m, v_0 p'_m) \\ &= (-1)^m d(u_1 p'_m, v_1 p'_m) = d(u_1, v_1) \end{aligned}$$

by Lemma 2.2, and

$$\begin{aligned} d(u_0, w) &= (-1)^{m+1} d(u_0 q'_m, wq'_m) \\ &= (-1)^{m+1} d(u_2 q'_m, v_2 q'_m) = d(u_2, v_2) \end{aligned}$$

by Lemma 2.3. Hence Theorem 2.4 follows from the addition formula for separation elements.

3. Mappings of complexes

Let K be a CW-complex and let L be a subcomplex. Let (M, N) denote the pair obtained from (J, K) by pinching L to a point, where J is the cone on base K. We describe L as *retractile* in K if N is contractible in M. If L is retractile in K, the homology exact sequence reduces to

$$0 \to H(L) \to H(K) \to H(K, L) \to 0$$

although it need not split as it does when L is a retract of K. We prove the lemma [cf. (2.7) of (**15**)]:

LEMMA 3.1. *Let $L_1,\ldots, L_n$ be subcomplexes of K each of which is a retract of K. Suppose that there exist retractions $r_i: K \to L_i$ $(i = 1,\ldots, n)$ with $r_i L_j \subset L_j$ $(j = 1,\ldots, n)$. Then L is retractile in K, where $L = L_1 \cup \ldots \cup L_n$.*

For consider the subcomplexes

$$L'_0 \subset L'_1 \subset \ldots \subset L'_n = L,$$

where L'_0 is defined to be the basepoint and

$$L'_i = L_1 \cup \ldots \cup L_i \quad (i = 1,\ldots, n).$$

The composition of the inclusion $K \to J$ with the projection $J \to M$ is a nul-homotopic map $f: K \to M$ which is constant on L. Make the inductive hypothesis that f is nul-homotopic (rel L'_{i-1}), where $1 \leqslant i \leqslant n$. If $g_t: K \to X$ is one such nul-homotopy (rel L'_{i-1}), then another is given by

$$h_t = \begin{cases} g_{2t} & (0 \leqslant t \leqslant \frac{1}{2}), \\ g_{2-2t}\, r_i & (\frac{1}{2} \leqslant t \leqslant 1). \end{cases}$$

Since $h_t = h_{1-t}$ on L_i, it follows that $h_t | L'_i$ can be deformed into the trivial homotopy and hence, by the homotopy-extension theorem, that h_t can be deformed into a nul-homotopy of f (rel L'_i). Therefore f is nul-homotopic (rel L), by induction, and so N is contractible in M. This proves Lemma 3.1.

For example, suppose that $K = K_1 \times \ldots \times K_r$, where $K_1,\ldots, K_r$ are CW-complexes with basepoints $e_1,\ldots, e_r$. Take the L_i to be the $2^r - 2$ *proper* subcomplexes $K'_1 \times \ldots \times K'_r$, where $K'_1 = e_1$ or K_1, ..., $K'_r = e_r$ or K_r. Then the conditions of Lemma 3.1 are satisfied, and so L is retractile in K, where L is determined by

$$K - L = (K_1 - e_1) \times \ldots \times (K_r - e_r).$$

In the applications we take $K_1,\ldots, K_r$ to be spheres of various dimensions, as in § 3 of (**15**).

Let X be a space. We now prove the lemma:

LEMMA 3.2. *Let L be retractile in K. Let $f, f' : K \to X$ be nul-homotopic maps which agree on L. Then $f \simeq f'$ (rel L).*

Let f_t be a nul-homotopy of f. By the homotopy-extension theorem there exists a homotopy f'_t of f' into f'', say, such that $f_t|L = f'_t|L$. Since f'' is a nul-homotopic map which is constant on L and since L is retractile in K, it follows that f'' is nul-homotopic (rel L). Let f''_t be a homotopy (rel L) of f'' into the constant map. Let g_t denote the

homotopy of f into f' which is defined by

$$g_t = \begin{cases} f_{3t} & (0 \leqslant t \leqslant \frac{1}{3}), \\ f''_{2-3t} & (\frac{1}{3} \leqslant t \leqslant \frac{2}{3}), \\ f'_{3-3t} & (\frac{2}{3} \leqslant t \leqslant 1). \end{cases}$$

Since $g_t = g_{1-t}$ on L, it follows that $g_t|L$ can be deformed into the constant homotopy and hence, by the homotopy-extension theorem, g_t can be deformed into a homotopy of f into $f'(\text{rel } L)$. This proves Lemma 3.2.

In the following two lemmas we consider a map $p:X \to Y$, where X and Y are spaces, which induces a $(1, 1)$-correspondence

$$p_*:\pi_n(X) \to \pi_n(Y) \quad (n = 0, 1, \ldots).$$

Let K be a CW-complex and L a subcomplex.

LEMMA 3.3. *Let $f:K \to Y$ and $g:L \to X$ be maps such that $pg = f|L$. Then there exists a map $h:K \to X$, extending g, such that $ph \simeq f$* (rel L).

Let Z denote the mapping cylinder of p. Then $ri = p$ and $rj = 1$, as shown below, where i, j are inclusions and r is the standard retraction.

$$X \xrightarrow{i} Z \xrightarrow{r} Y \xrightarrow{j} Z$$

Moreover $jr \simeq 1$ (rel Y), by a deformation $u_t: Z \to Z$ such that $ru_t = r$ for $0 \leqslant t \leqslant 1$. Write $ig = g':L \to Z$, $jf = f':K \to Z$. Since $u_t g'$ is a deformation of $f'|L$ into g', it follows from the homotopy-extension theorem that $f' \simeq f''$, say, where f'' is an extension of g' such that $rf' \simeq rf''$ (rel L). However $\pi_n(Z, X) = 0$ since r is a homotopy equivalence and $p = ri$. Therefore $f'' \simeq ih$ (rel L), by (K) [(**16**) 228], where $h:K \to X$ is a map such that $g = h|L$. Hence

$$f = rjf = rf' \simeq rf'' \simeq rih = ph \quad (\text{rel } L),$$

which proves the lemma.

Hence we deduce the lemma:

LEMMA 3.4. *Let $h_0, h_1:K \to X$ be maps which agree on L such that $ph_0 \simeq ph_1$ (rel L). Then $h_0 \simeq h_1$ (rel L).*

For let† $g:K \times I \cup L \times \dot{I} \to X$ be defined by

$$g(x, t) = \begin{cases} h_t(x) & (x \in K,\ t \in \dot{I}), \\ h_0(x) = h_1(x) & (x \in L,\ t \in I). \end{cases}$$

By hypothesis pg can be extended over $K \times I$. Hence g can be extended, by Lemma 3.3, and this provides a homotopy of h_0 into h_1 (rel I).

† Here I denotes the unit interval, where $0 \leqslant t \leqslant 1$.

4. Proof of Theorem 1.1

Let A be an H-space with basepoint a_0. Consider a map $u: K \to A$, where K is a CW-complex, and write $v = u | L$, where L is a subcomplex. We prove the theorem:

THEOREM 4.1. *Let $f: K \to A$ and $g: L \to A$ be maps such that $g.v = f | L$. Then there exists a map $h: K \to A$, extending g, such that $h.u \simeq f$* (rel L).

We apply Lemma 3.3 with $X = Y = B$, where $B = A \times A$, and with $p: B \to B$ defined by

$$p(x, y) = (x.y, y) \quad (x, y \in A).$$

It is shown in § 4 of (**10**) that p induces an automorphism of the homotopy groups of B. We have $pg' = f' | L$, where

$$f' = (f, u): K \to B, \qquad g' = (g, v): L \to B.$$

Hence there exists a map $h': K \to B$, extending g', such that $ph' \simeq f'$ (rel L). Consider the maps h, w: $K \to A$ defined by $h' = (h, w)$. We have $g = h | L$, $v = w | L$. Furthermore $h.w \simeq f$ (rel L), and $w \simeq u$ (rel L), so that $h.u \simeq f$ (rel L). This proves Theorem 4.1.

Similarly we obtain the theorem:

THEOREM 4.2. *Let $f: K \to A$ and $g: L \to A$ be maps such that $v.g = f | L$. Then there exists a map $h: K \to A$, extending g, such that $u.h \simeq f$* (rel L).

By applying Lemma 3.4 instead of Lemma 3.3, we obtain the theorem:

THEOREM 4.3. *Let h_0, $h_1: K \to A$ be maps which agree on L such that either $h_0.u \simeq h_1.u$* (rel L) *or $u.h_0 \simeq u.h_1$* (rel L). *Then $h_0 \simeq h_1$* (rel L).

If L is the basepoint, these theorems show that $\pi(K; A)$ is a loop, which proves (1.1). If L is retractile in K, we deduce the corollary:

COROLLARY 4.4. *Let L be retractile in K. Let $f, f': K \to A$ be homotopic maps which agree on L. Then $f \simeq f'$* (rel L).

For by Theorem 1.1 there exists a map $u: K \to A$ such that $f.u$ and $f'.u$ are nul-homotopic. Hence $f.u \simeq f'.u$ by Lemma 3.2, and so Corollary 4.4 follows from Theorem 4.3.

5. Maps into an H-space

Let K be a CW-complex with a subcomplex L such that $e^n \cup L = K$, as in § 2. Let A be an H-space. If u_i, $v_i: K \to A$ $(i = 0, 1)$ are maps such that $u_i | L = v_i | L$, then

$$d(u_0.v_0, u_1.v_1) = d(u_0, v_0) + d(u_1, v_1) \tag{5.1}$$

since the structures of $\pi_n(A)$ and $\pi(S^n; A)$ coincide.† This relation enables us to discuss separation elements as follows.

† Cf. Theorem 10.1 below. Because of this we generally use the additive notation for $\pi(S^n; A)$.

Let u, $v:K \to A$ be maps which agree on L. By Theorem 1.1 there exists a map $w:K \to A$ such that $v.w$ is nul-homotopic. Let h_t be a homotopy of $v.w$ into the constant map and let g_t be a homotopy of $u.w$ into k, say, where g_t agrees with h_t on L. Since k is constant on L, there is an induced homomorphism

$$k_*:\pi_n(K, L) \to \pi_n(A).$$

Since $d(g_t, h_t)$ does not vary with t, we have

$$d(u.w, v.w) = d(g_1, h_1) = k_* \sigma \in \pi_n(A),$$

where $\sigma \in \pi_n(K, L)$ is the class of the characteristic map. However,

$$d(u.w, v.w) = d(u, v)+d(w, w) = d(u, v),$$

by (5.1), and so

$$d(u, v) = k_* \sigma. \tag{5.2}$$

For example, suppose that the multiplication on A admits a continuous right-inverse. Then we may take k to be the product of u with the right-inverse of v, in which case (5.2) provides an appropriate way of defining $d(u, v)$. Of course left-translation instead of right-translation would do just as well.

Let K' be a CW-complex with a subcomplex L' whose complement is an m-cell e^m. Let $f:K' \to K$ be a map such that $fL' \subset L$. By identifying L' to a point in K' and L to a point in K we obtain from f a map $g:S^m \to S^n$ of class θ, say, where $\theta \in \pi_m(S^n)$. Let $\sigma' \in \pi_m(K', L')$, $\sigma \in \pi_n(K, L)$ denote the elements represented by the characteristic maps. Let u, v, $k:K \to A$ be as in (5.2). Then

$$(k_* \sigma) \circ \theta = (kf)_* \sigma',$$

by taking representatives, and hence, by (5.2),

$$d(u, v) \circ \theta = d(uf, vf). \tag{5.3}$$

We conclude this section by using Corollary 4.4 to prove the theorem

THEOREM 5.4. *Let L be retractile in K. Let $u_i, v_i:K \to A$ $(i = 0, 1)$ be maps such that $u_i|L = v_i|L$. If $u_0 \simeq u_1$ and $v_0 \simeq v_1$ then*

$$d(u_0, v_0) = d(u_1, v_1).$$

Let u_t be a deformation of u_0 into u_1. By the homotopy-extension theorem there exists a deformation h_t of v_0 into v_1', say, such that $h_t|L = u_t|L$. By (4.4) $v_1 \simeq v_1'$ (rel L) since v_1 and v_1' are homotopic maps which agree on L. Therefore

$$d(u_1, v_1) = d(u_1, v_1') = d(u_0, v_0)$$

since $d(u_t, h_t)$ does not vary with t. This proves Theorem 5.4.

Let $u, v: K \to A$ be maps such that $u|L \simeq v|L$. We write

$$\delta(u, v) = d(u, v'),$$

where v' is any map which is homotopic to v and agrees with u on L. It follows from Corollary 4.4 that $\delta(u, v)$ is independent of the choice of v', and that $u \simeq v$ if and only if $\delta(u, v) = 0$. It should be emphasized that this way of extending the definition of separation elements is based on the hypothesis that L is retractile in K, as well as that A is an H-space.

6. Obstruction to homotopy-commutativity

Let A be an H-space. Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$ be homotopy classes of maps

$$u: S^p \to A, \qquad v: S^q \to A,$$

respectively. We write $\langle \alpha, \beta \rangle = d(f, g) \in \pi_{p+q}(A)$, where $f, g: S^p \times S^q \to A$ are the maps given on $x \in S^p$, $y \in S^q$ by

$$f(x, y) = \xi . \eta, \qquad g(x, y) = \eta . \xi \quad (\xi = ux,\ \eta = vy).$$

If the multiplication on A is homotopy-commutative, then $f \simeq g$ and so $\langle \alpha, \beta \rangle = 0$, by Corollary 4.4. Thus $\langle \alpha, \beta \rangle$ can be regarded as an obstruction to homotopy-commutativity.

We prove that $(\alpha, \beta) \to \langle \alpha, \beta \rangle$ is a bilinear operation and therefore induces a homomorphism of the tensor product

$$\pi_p(A) \otimes \pi_q(A) \to \pi_{p+q}(A).$$

Suppose that $\alpha = \alpha' + \alpha''$, where $\alpha', \alpha'' \in \pi_p(A)$. Let $u', u'': S^p \to A$ be representatives of α', α'' such that u' is constant on E^p_- and u'' is constant on E^p_+. Let u, the representative of α, map E^p_+ according to u' and E^p according to u''. By substituting u' and u'' for u in the definition of (f, g) we obtain further pairs of maps (f', g') and (f'', g'') such that

$$\langle \alpha', \beta \rangle = d(f', g'), \qquad \langle \alpha'', \beta \rangle = d(f'', g'').$$

In Theorem 2.4, with $m = p$, we take

$$u_0 = f,\ v_0 = g; \quad u_1 = f',\ v_1 = g'; \quad u_2 = f'',\ v_2 = g''.$$

The hypotheses of the theorem are verified and so

$$d(f, g) = d(f', g') + d(f'', g''),$$

which shows that the operation is left-linear. Right-linearity can be proved similarly, or else can be deduced from left-linearity by use of the following commutation law.

Let T: $S^q \times S^p \to S^p \times S^q$ be defined by $T(y, x) = (x, y)$. Then $d(Tf, Tg) = (-1)^{pq} d(f, g)$ since the degree of T is $(-1)^{pq}$, and hence

$$\langle \alpha, \beta \rangle = (-1)^{pq+1} \langle \beta, \alpha \rangle. \tag{6.1}$$

THEOREM 6.2. *Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_r(A)$. If the multiplication on A is homotopy-associative then*

$$(-1)^{rp}\langle\langle\alpha,\beta\rangle,\gamma\rangle+(-1)^{pq}\langle\langle\beta,\gamma\rangle,\alpha\rangle+(-1)^{qr}\langle\langle\gamma,\alpha\rangle,\beta\rangle = 0.$$

The relation in Theorem 6.2 is known as the *Jacobi identity*. The proof in § 4 of (**15**) is for group-like spaces but can readily be adapted since it is the group-structure of $\pi(K;A)$ which is needed and this is provided by Corollary 1.2 above.

We turn to consideration of naturality. Let $\lambda \in \pi_i(S^p)$, $\mu \in \pi_j(S^q)$. Then

$$\langle\alpha \circ \lambda, \beta \circ \mu\rangle = \langle\alpha,\beta\rangle \circ (\lambda \divideontimes \mu), \tag{6.3}$$

by (5.3), where $\lambda \divideontimes \mu \in \pi_{i+j}(S^{p+q})$ denotes the reduced join as defined in § 3 of (**2**).

Let $h: A \to A'$ be a map, where A and A' are H-spaces, and let $k, l: A \times A \to A'$ denote the maps defined by

$$k(x,y) = (hx).(hy), \qquad l(x,y) = h(x.y) \quad (x,y \in A).$$

We say that 'h is an H-map' if $k \simeq l$, in which case h induces a homomorphism

$$h_*: \pi(X;A) \to \pi(X;A').$$

For example, suppose that A and A' are the same space, with multiplications m and m'. Then m is equivalent to m' if and only if the identity transformation is an H-map.

Let $h: A \to A'$ be an H-map. We conclude this section by proving that

$$h_*\langle\alpha,\beta\rangle = \langle h_*\alpha, h_*\beta\rangle, \tag{6.4}$$

which incidentally shows that $\langle\alpha,\beta\rangle$ is unchanged when the multiplication on A is altered to an equivalent one. Let f, g: $S^p \times S^q \to A$ mean the same as before, and let f', g': $S^p \times S^q \to A'$ be similarly defined with hu, hv in place of u, v. We have $hf \simeq f'$ and $hg \simeq g'$ since h is an H-map. By Theorem 5.4 and naturality,

$$d(f',g') = d(hf,hg) = h_* d(f,g),$$

which proves (6.4).

7. Obstruction to homotopy-associativity

Let A be an H-space. Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_r(A)$ be homotopy classes of maps u: $S^p \to A$, v: $S^q \to A$, w: $S^r \to A$, respectively. We write $\langle\alpha,\beta,\gamma\rangle = d(f,g) \in \pi_{p+q+r}(A)$, where f, g: $S^p \times S^q \times S^r \to A$ are the maps given on $x \in S^p$, $y \in S^q$, $z \in S^r$ by

$$f(x,y,z) = (\xi.\eta).\zeta, \qquad g(x,y,z) = \xi.(\eta.\zeta)$$
$$(\xi = ux,\ \eta = vy,\ \zeta = wz).$$

If the multiplication on A is homotopy-associative, then $f \simeq g$, and so $\langle\alpha,\beta,\gamma\rangle = 0$, by Corollary 4.4. Thus $\langle\alpha,\beta,\gamma\rangle$ can be regarded as an obstruction to homotopy-associativity.

We prove that $(\alpha,\beta,\gamma) \to \langle\alpha,\beta,\gamma\rangle$ is a trilinear operation and therefore induces a homomorphism of the tensor product

$$\pi_p(A) \otimes \pi_q(A) \otimes \pi_r(A) \to \pi_{p+q+r}(A).$$

Suppose that $\alpha = \alpha'+\alpha''$, where α', $\alpha'' \in \pi_p(A)$. Let u', u'': $S^p \to A$ be representatives of α', α'' such that u' is constant on E^p_- and u'' is constant on E^p_+. Let u, the representative of α, map E^p_+ according to u' and E^p_- according to u''. By substituting u' and u'' for u in the definition of (f,g) we obtain further pairs of maps (f',g') and (f'',g'') such that

$$\langle\alpha',\beta,\gamma\rangle = d(f',g'), \qquad \langle\alpha'',\beta,\gamma\rangle = d(f'',g'').$$

We apply Theorem 2.4 with $m = p$, as before, and obtain that

$$d(f,g) = d(f',g')+d(f'',g''),$$

which proves the operation to be linear in α. Linearity in β and γ is proved similarly.

THEOREM 7.1. *Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_r(A)$. If the multiplication on A is homotopy-commutative, then*

(*a*) $\langle\alpha,\beta,\gamma\rangle+(-1)^{rp+pq+qr}\langle\gamma,\beta,\alpha\rangle = 0$,

(*b*) $(-1)^{rp}\langle\alpha,\beta,\gamma\rangle+(-1)^{pq}\langle\beta,\gamma,\alpha\rangle+(-1)^{qr}\langle\gamma,\alpha,\beta\rangle = 0$.

Let T: $S^r\times S^q\times S^p \to S^p\times S^q\times S^r$ be defined by $T(z,y,x) = (x,y,z)$. Let f, g be as before and let $\bar{f}$, $\bar{g}$: $S^p\times S^q\times S^r \to A$ be defined by

$$\bar{f}(x,y,z) = (\zeta.\eta).\xi, \qquad \bar{g}(x,y,z) = \zeta.(\eta.\xi).$$

Write $t = (-1)^{rp+pq+qr}$, which is the degree of T. Then

$$td(\bar{f},\bar{g}) = d(\bar{f}T,\bar{g}T) = \langle\gamma,\beta,\alpha\rangle,$$

and so

$$\begin{aligned} d(f.\bar{f},\, g.\bar{g}) &= d(f,g)+d(\bar{f},\bar{g}), \quad \text{by (5.1)}, \\ &= \langle\alpha,\beta,\gamma\rangle+t\langle\gamma,\beta,\alpha\rangle. \end{aligned}$$

If the multiplication on A is homotopy-commutative, then $f.\bar{f} \simeq g.\bar{g}$, and so $d(f.\bar{f},\, g.\bar{g}) = 0$, by Corollary 4.4. This proves (*a*).

To prove (*b*) we introduce the maps f', g'; f'', g'': $S^p\times S^q\times S^r \to A$ which are given by

$$f'(x,y,z) = (\eta.\zeta).\xi, \qquad g'(x,y,z) = \eta.(\zeta.\xi),$$

$$f''(x,y,z) = (\zeta.\xi).\eta, \qquad g''(x,y,z) = \zeta.(\xi.\eta).$$

By rearranging the factors of $S^p\times S^q\times S^r$, as before, we find that

$$d(f',g') = (-1)^{pq+pr}\langle\beta,\gamma,\alpha\rangle, \qquad d(f'',g'') = (-1)^{pr+qr}\langle\gamma,\alpha,\beta\rangle.$$

If the multiplication is homotopy-commutative, then

$$f \simeq g'', \quad f' \simeq g, \quad f'' \simeq g',$$

and therefore $\qquad f.f' \simeq g''.g, \qquad g.g' \simeq f''.g.$

Hence $\qquad d(f.f', g.g') = d(g''.g, f''.g),$

by Theorem 5.4, and so it follows from (5.1) and the addition law that

$$d(f,g)+d(f',g')+d(f'',g'') = 0.$$

This proves (*b*) and completes the proof of Theorem 7.1.

Finally we discuss naturality. Let $\lambda \in \pi_i(S^p)$, $\mu \in \pi_j(S^q)$, $\nu \in \pi_k(S^r)$. The reduced join is associative so that

$$\lambda \divideontimes \mu \divideontimes \nu \in \pi_{i+j+k}(S^{p+q+r})$$

is defined, and by (5.3) we have

$$\langle \alpha \circ \lambda, \beta \circ \mu, \gamma \circ \nu \rangle = \langle \alpha, \beta, \gamma \rangle \circ (\lambda \divideontimes \mu \divideontimes \nu). \tag{7.2}$$

Let $h\colon A \to A'$ be an H-map, where A and A' are H-spaces. We prove that

$$h_*\langle \alpha, \beta, \gamma \rangle = \langle h_*\alpha, h_*\beta, h_*\gamma \rangle, \tag{7.3}$$

which incidentally shows that $\langle \alpha, \beta, \gamma \rangle$ is unchanged when the multiplication on A is altered to an equivalent one. Let f, g mean the same as before and let $f', g'\colon S^p \times S^q \times S^r \to A'$ be similarly defined with hu, hv, hw in place of u, v, w. Then $f' \simeq hf$ and $g' \simeq hg$ since h is an H-map. By Theorem 5.4 and naturality of separation elements,

$$d(f', g') = d(hf, hg) = h_* d(f, g),$$

which proves (7.3).

8. Relation between the two obstructions

Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_r(A)$, where A is an H-space. We consider the expressions

$$\theta(\alpha,\beta,\gamma) \equiv (-1)^{r(p+1)}\langle \alpha,\beta,\gamma\rangle + (-1)^{p(q+1)}\langle \beta,\gamma,\alpha\rangle + (-1)^{q(r+1)}\langle \gamma,\alpha,\beta\rangle,$$

$$\phi(\alpha,\beta,\gamma) \equiv (-1)^{p(r+1)}\langle \gamma,\beta,\alpha\rangle + (-1)^{q(p+1)}\langle \alpha,\gamma,\beta\rangle + (-1)^{r(q+1)}\langle \beta,\alpha,\gamma\rangle,$$

$$\psi(\alpha,\beta,\gamma) \equiv (-1)^{rp}\langle\langle \alpha,\beta\rangle,\gamma\rangle + (-1)^{pq}\langle\langle \beta,\gamma\rangle,\alpha\rangle + (-1)^{qr}\langle\langle \gamma,\alpha\rangle,\beta\rangle,$$

and show that they satisfy the relation

$$\theta(\alpha,\beta,\gamma) = (-1)^{rp+pq+qr}\phi(\alpha,\beta,\gamma) + (-1)^{p+q+r}\psi(\alpha,\beta,\gamma). \tag{8.1}$$

The proof is based on Theorem 6.2, the Jacobi identity, to which (8.1) reduces when the multiplication is homotopy-associative. We make use of the theory of reduced-product spaces [see (**3**)] which generally facilitates the discussion of non-associative H-spaces.

Consider a space X and an H-space A. Let X_∞, the reduced-product space of X, be constructed as in (**3**). Points of X_∞, we recall, are represented by finite sequences of points in X, subject to certain identifications. A subspace X_m ($m = 0, 1,...$) is formed by the sequences with not more than m terms. Let $[x_1,..., x_m]$ denote the point of X_m represented by the sequence $(x_1,..., x_m)$, where $x_1,..., x_m \in X$. We identify $x \in X$ with $[x] \in X_1$, so that $X \subset X_\infty$. Each map $h: X \to A$ admits an extension $h': X_\infty \to A$ which is defined inductively by

$$h'[x_1, x_2,..., x_m] = h(x_1).h'[x_2,..., x_m] \quad (m = 2, 3,...).$$

Thus the homomorphism of homotopy groups induced by h can be decomposed into

$$\pi_n(X) \xrightarrow{\sigma} \pi_n(X_\infty) \xrightarrow{\rho} \pi_n(A),$$

where σ denotes the injection and ρ the homomorphism induced by h'.

Suppose now that $X = K$, a countable CW-complex with only one 0-cell. By (1.11) of (**3**), X_∞ is then an H-space with an associative multiplication such that $[x_1,..., x_m] = x_1 . ... . x_m$. Notice that

$$h'(x_1 . x_2 . x_3) = y_1 . (y_2 . y_3), \tag{8.2}$$

where $y_i = hx_i$ ($i = 1, 2, 3$). Since $h'(x_1 . x_2) = y_1 . y_2$, it follows that

$$\rho\langle\lambda, \mu\rangle = \langle\rho\lambda, \rho\mu\rangle, \tag{8.3}$$

where $\lambda \in \sigma\pi_p(X)$, $\mu \in \sigma\pi_q(X)$. When the multiplication on A is homotopy-associative, we have naturality with respect to ρ, as in (6.4), since h' is an H-map. In general this will not be the case, and the following lemma shows how the deviation from naturality is related to the obstruction to homotopy-associativity:

LEMMA 8.4. *Let* $\alpha = \rho\lambda$, $\beta = \rho\mu$, $\gamma = \rho\nu$, *where* $\lambda \in \sigma\pi_p(X)$, $\mu \in \sigma\pi_q(X)$ $\nu \in \sigma\pi_r(X)$. *Then*

$$\langle\langle\alpha, \beta\rangle, \gamma\rangle - \rho\langle\langle\lambda, \mu\rangle, \nu\rangle = (-1)^{p+q}(\langle\alpha, \beta, \gamma\rangle - (-1)^{pq}\langle\beta, \alpha, \gamma\rangle).$$

By hypothesis λ, μ, ν can be represented by

$$u: S^p \to X, \qquad v: S^q \to X, \qquad w: S^r \to X,$$

respectively, regarded as maps into X_∞. Write $p+q = m$ and consider the projections

$$p'_m, q'_m: V^m \times S^r \to S^m \times S^r,$$

as in § 2, which are given by

$$p'_m(t, z) = (p_m t, z), \qquad q'_m(t, z) = (q_m t, z) \quad (t \in V^m, z \in S^r).$$

Let $k: V^m \to S^p \times S^q$ be an orientation-preserving map, topological on the interior of V^m, such that $kS^{m-1} \subset S^p \times e \cup e \times S^q$. Then $kt = (x, y)$, say, where $x \in S^p$, $y \in S^q$. Write $\xi = ux$, $\eta = vy$, $\zeta = wz$, and $s = (t, z)$.

By (2.1), $\langle\langle\lambda,\mu\rangle,\nu\rangle = d(f,g)$, where f, g: $S^m\times S^r \to X_\infty$ are the maps given by

$$fp'_m(s) = \xi.\eta.\zeta, \qquad fq'_m(s) = \eta.\xi.\zeta,$$
$$gp'_m(s) = \zeta.\xi.\eta, \qquad gq'_m(s) = \zeta.\eta.\xi.$$

Write $f' = h'f$, $g' = h'g$, so that

$$d(f',g') = \rho d(f,g) = \rho\langle\langle\lambda,\mu\rangle,\nu\rangle. \tag{8.5}$$

By (8.2), f', g': $S^m\times S^r \to A$ are given by

$$f'p'_m(s) = \xi'.(\eta'.\zeta'), \qquad f'q'_m(s) = \eta'.(\xi'.\zeta'),$$
$$g'p'_m(s) = \zeta'.(\xi'.\eta'), \qquad g'q'_m(s) = \zeta'.(\eta'.\xi'),$$

where $\xi' = h\xi$, $\eta' = h\eta$, $\zeta' = h\zeta$. Consider also the maps

$$f'', g'': S^m\times S^r \to A$$

which are given by

$$f''p'_m(s) = (\xi'.\eta').\zeta', \qquad f''q'_m(s) = (\eta'.\xi').\zeta',$$
$$g''p'_m(s) = \xi'.(\eta'.\zeta'), \qquad g''q'_m(s) = (\eta'.\xi').\zeta'.$$

By (2.1) and the addition formula,

$$\langle\rho\langle\lambda,\mu\rangle,\rho\nu\rangle = d(f'',g') = d(f'',f')+d(f',g').$$

Hence it follows from (8.3) and (8.5) that

$$\langle\langle\alpha,\beta\rangle,\gamma\rangle-\rho\langle\langle\lambda,\mu\rangle,\nu\rangle = d(f'',f'). \tag{8.6}$$

The product of k with the identity on S^r determines a relative homeomorphism

$$(V^m\times S^r,\ S^{m-1}\times S^r\cup V^m\times e)$$
$$\to (S^p\times S^q\times S^r,\ e\times S^q\times S^r\cup S^p\times e\times S^r\cup S^p\times S^q\times e),$$

which is compatible with orientations. Hence

$$\langle\alpha,\beta,\gamma\rangle = d(f''p'_m, g''p'_m) = (-1)^m d(f'',g''),$$

by Lemma 2.2, and moreover

$$(-1)^{pq}\langle\beta,\alpha,\gamma\rangle = d(g''q'_m, f'q'_m) = (-1)^{m+1}d(g'',f'),$$

by Lemma 2.3. Hence Lemma 8.4 is obtained from (8.6) and the addition formula

$$d(f'',g'')+d(g'',f') = d(f'',f').$$

Now we are ready to prove (8.1). In the lemma take $X = S^p\times S^q\times S^r$ and define h: $X\to A$ by multiplying representatives together so that

$$\alpha\in\rho\sigma\pi_p(X), \qquad \beta\in\rho\sigma\pi_q(X), \qquad \gamma\in\rho\sigma\pi_r(X).$$

Let $\lambda \in \sigma\pi_p(X)$, $\mu \in \sigma\pi_q(X)$, $\nu \in \sigma\pi_r(X)$ be elements such that $\alpha = \rho\lambda$, $\beta = \rho\mu$, $\gamma = \rho\nu$. Then $\psi(\lambda, \mu, \nu) = 0$, by Theorem 6.2, since the multiplication on X_∞ is associative. But

$$(-1)^{p+q+r}(\psi(\alpha, \beta, \gamma) - \rho\psi(\lambda, \mu, \nu)) = \theta(\alpha, \beta, \gamma) - (-1)^{rp+pq+qr}\phi(\alpha, \beta, \gamma),$$

by Lemma 8.4 and its permutations. Hence (8.1) follows at once.

9. Multiplication on spheres

The case $A = S^n$ ($n \geq 1$) exhibits a number of interesting features. Let n be odd, for otherwise S^n is not an H-space, and consider a multiplication on S^n. We write

$$\lambda_n = \langle \iota_n, \iota_n \rangle \in \pi_{2n}(S^n), \qquad \mu_n = \langle \iota_n, \iota_n, \iota_n \rangle \in \pi_{3n}(S^n),$$

where ι_n denotes the generator of $\pi_n(S^n)$ represented by the identity map. Notice that $\lambda_n = 0$ ($\mu_n = 0$) is the necessary and sufficient condition for the multiplication to be homotopy-commutative (homotopy-associative). By (6.3) and (7.2),

$$\left.\begin{aligned} &(a) \quad \langle \alpha, \beta \rangle = \lambda_n \circ (\alpha * \beta) \\ &(b) \quad \langle \alpha, \beta, \gamma \rangle = \mu_n \circ (\alpha * \beta * \gamma) \end{aligned}\right\}, \tag{9.1}$$

where $\alpha \in \pi_p(S^n)$, $\beta \in \pi_q(S^n)$, $\gamma \in \pi_r(S^n)$. Thus the order of $\langle \alpha, \beta \rangle$ is a divisor of the order of λ_n, and the order of $\langle \alpha, \beta, \gamma \rangle$ is a divisor of the order of μ_n. From (9.1 (b)) and the commutative law for the reduced join, as in (3.10) of (**2**), we obtain the relations

$$\left.\begin{aligned} &(a) \quad \langle \alpha, \beta, \gamma \rangle + (-1)^{rp+pq+qr}\langle \gamma, \beta, \alpha \rangle = 0 \\ &(b) \quad (-1)^{rp}\langle \alpha, \beta, \gamma \rangle = (-1)^{pq}\langle \beta, \gamma, \alpha \rangle = (-1)^{qr}\langle \gamma, \alpha, \beta \rangle \end{aligned}\right\}. \tag{9.2}$$

Notice that (9.2 (a)) is the same relation as occurs in Theorem 7.1 (a) although the multiplication may not be homotopy-commutative. We take up this point in Corollary 10.3 below. It follows from (9.1 (a)) that

$$\langle \langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle \rangle = \langle \langle \langle \alpha, \beta \rangle, \gamma \rangle, \delta \rangle, \tag{9.3}$$

where $\delta \in \pi_s(S^n)$. We have $2\langle \lambda_n, \lambda_n \rangle = 0$, by (6.1), but $\langle \lambda_n, \lambda_n \rangle \neq 0$ in the case of quaternionic multiplication on S^3. Probably only a few non-trivial elements can be built up from ι_n by the two kinds of bracketing operation, but the investigation of such matters seems to demand more information about the homotopy groups of spheres than is at present available.

It is interesting to determine the orders of λ_n and μ_n. To avoid trivialities we exclude the case $n = 1$. Let $n = 3$ or 7 since it is shown in (**1**) that other values of n are impossible. We represent points of

S^n by unit quaternions when $n = 3$, by unit Cayley numbers when $n = 7$, so that the pure imaginaries represent points of S^{n-1}. We consider first the classical multiplication on S^n, i.e. the quaternionic when $n = 3$, the Cayley when $n = 7$. By (5.1), $\lambda_n = d(u, v)$, where

$$u, v \colon S^n \times S^n \to S^n$$

are the maps given by

$$u(x, y) = x . y . x^{-1}, \qquad v(x, y) = y \quad (x, y \in S^n).$$

If y is pure-imaginary, then so is $u(x, y)$, and the transformation is a rotation $\rho(x)$ of S^{n-1}. By definition $d(u, v)$ is the element obtained from $\rho \colon S^n \to R_n$ by the Hopf construction, where R_n is the rotation group of S^{n-1}. Hence $\lambda_n = J\kappa_n$, where $\kappa_n \in \pi_n(R_n)$ denotes the class of ρ and where

$$J \colon \pi_n(R_n) \to \pi_{2n}(S^n)$$

denotes the homomorphism which is defined by means of the Hopf construction. Recall† that $\pi_n(R_n)$ is cyclic infinite and that $\pi_{2n}(S^n)$ is cyclic of order 12 ($n = 3$), 120 ($n = 7$). The generalized Hopf invariant of λ_n is non-zero‡ and, since $2\pi_{2n}(S^{2n-1}) = 0$, this shows that the image of J has an odd index. Also it follows from Theorem 2 of (**8**) that the image contains elements of order 3 ($n = 3$), 15 ($n = 7$). Therefore J is onto, and moreover λ_n generates $\pi_{2n}(S^n)$ since κ_n generates $\pi_n(R_n)$.

Now change the classical multiplication m to an arbitrary multiplication m'. Invariants of m' are distinguished from those of m by adding a prime in the notation. Since $\pi_{2n}(S^n)$ is cyclic, we have $d(m, m) = r\lambda_n$, where r is an integer, and a convenient m' with this property can be defined in terms of the classical multiplication by

$$m'(x, y) = x^r . x . y . x^{-r} \quad (x, y \in S^n).$$

Notice that every left translation is a rotation. Since n is odd, we have $\lambda'_n - \lambda_n = 2d(m', m)$, by the addition formula for separation elements, and so

$$\lambda'_n = (2r+1)\lambda_n. \tag{9.4}$$

In the case of μ'_n the situation is more complicated. A complete determination of μ'_3 is given in § 3 of (**7**). A similar argument for S^7 shows that the order of μ'_7 is divisible by 3, which is the greatest odd divisor of the order of $\pi_{21}(S^7)$. By (8.1) we have

$$6\mu'_n = 3\langle \lambda'_n, i_n \rangle' = 3(2r+1)^2 \lambda_n \circ E^n\lambda_n$$

by (9.1 (a)) and (9.4), where $E^n\lambda_n = \lambda_n \circledast \iota_n$. According to (**12**),

$$24\pi_{21}(S^7) = 0,$$

† See (**11**), (**12**), (**13**) for information about these groups.
‡ See § 4 of (**11**) and § 8 of (**14**).

and the order of $\lambda_7 \circ E^7\lambda_7$ is divisible by 4, from which it follows easily that the order of μ_7' is 24.

In conclusion let us consider briefly the real projective n-space P^n, where $n > 1$. If $n = 3$ or 7, then m', in the above form, determines a multiplication on P^n. There are at least 12 classes of multiplication on P^3 and at least 120 on P^7. There are no homotopy-associative multiplications on P^7, and no homotopy-commutative multiplications on P^3 or P^7. There are no multiplications on P^n unless $n = 3$ or 7. All these statements follow from the corresponding results for multiplication on spheres [see (**1**), (**6**), (**7**)] with the aid of the covering homotopy theorem.

10. Appendix

Let Y be a space with basepoint y_0. Let SY denote the suspension of Y which is obtained from $Y \times I$ by identifying $Y \times \dot{I} \cup y_0 \times I$ with y_1, the basepoint in SY. Points of SY are represented by pairs (y, t) $(y \in Y,\ t \in I)$ subject to appropriate identifications. Let $R\colon SY \to SY$ denote the reflection map given by $R(y,t) = R(y, 1-t)$. Let

$$P,\ Q\colon SY \to SY$$

denote the maps given by

$$\begin{aligned} P(y,t) &= (y, 2t), & Q(y,t) &= z_0 & (t \leqslant \tfrac{1}{2}),\\ P(y,t) &= z_0, & Q(y,t) &= (y, 2t-1) & (t \geqslant \tfrac{1}{2}). \end{aligned}$$

Then $P \simeq 1 \simeq Q$.

Let X be a space and let A be an H-space. We prove the well-known theorem:

THEOREM 10.1. *If X is a suspension, then $\pi(X; A)$ forms an abelian group in which the inverse is given by reflection.*

Since X is a suspension, the maps P, Q, R: $X \to X$ are defined. Consider maps u, v, w: $X \to A$. We have

$$u.v \simeq (uP).(vQ) = (vQ).(uP) \simeq v.u,$$

which proves commutativity, and

$$(u.v).w \simeq ((uP).v).(wQ) = (uP).(v.(wQ)) \simeq u.(v.w),$$

which proves associativity. Since $u.(uR)$ and $(uR).u$ are nul-homotopic the proof of the theorem is complete.

Let M be a set with a binary operation, written multiplicatively, and a two-sided identity $1 \in M$. We say that M forms an *alternative* loop

if the following conditions are satisfied. First, every element $x \in M$ admits a two-sided inverse $x^{-1} \in M$. We describe a pair of elements as *involutory* if they are equal or if one is inverse to the other. The second condition is that

$$(x.y).z = x.(y.z) \quad (x, y, z \in M)$$

provided that one of the pairs (x, y), (x, z), (y, z) is involutory. Thus any two elements of an alternative loop generate a group. We prove the theorem:

THEOREM 10.2. *If A is a suspension, then $\pi(X; A)$ forms an alternative loop in which the inverse is given by reflection.*

Since A is a suspension, the maps P, Q, R: $A \to A$ are defined. If u: $X \to A$ is a map, then $u.(Ru)$ and $(Ru).u$ are nul-homotopic, so that the first condition is satisfied. In the case of the second condition x, y, $z \in \pi(X; A)$ can be represented by maps u, v, w: $X \to A$ which satisfy one of the six relations:

$$v = w, \quad v = Rw, \quad u = w, \quad u = Rw, \quad u = v, \quad u = Rv.$$

Define (u', v', w') to be one of the triples

$$\begin{array}{llll} (u, Pv, Qw) & (v = w), & (u, Pv, Pw) & (v = Rw); \\ (Pu, v, Qw) & (u = w), & (Pu, v, Pw) & (u = Rw); \\ (Pu, Qv, w) & (u = v), & (Pu, Pv, w) & (u = Rv). \end{array}$$

Then $\qquad (u.v).w \simeq (u'.v').w' = u'.(v'.w') \simeq u.(v.w),$

which establishes the second condition. This completes the proof of Theorem 10.2. Hence we deduce

COROLLARY 10.3. *Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$, $\gamma \in \pi_r(A)$. If A is a suspension, then*

$$\langle \alpha, \beta, \gamma \rangle + (-1)^{rp+pq+qr} \langle \gamma, \beta, \alpha \rangle = 0.$$

For let α, β, γ be represented by maps

$$u: S^p \to A, \quad v: S^q \to A, \quad w: S^r \to A,$$

respectively. Let f, g: $S^p \times S^q \times S^r \to A$ be the maps defined by

$$f(x, y, z) = (\xi.\eta).\zeta, \qquad g(x, y, z) = \xi.(\eta.\zeta),$$

where $x \in S^p$, $y \in S^q$, $z \in S^r$ and $\xi = ux$, $\eta = vy$, $\zeta = wz$. Write

$$\xi' = R\xi, \quad \eta' = R\eta, \quad \zeta' = R\zeta.$$

Since inversion is an anti-automorphism, by Theorem 10.2, it follows that $Rf \simeq \bar{f}$, $Rg \simeq \bar{g}$, where $\bar{f}$, $\bar{g}$ are the maps defined by

$$\bar{f}(x, y, z) = \zeta'.(\eta'.\xi'), \qquad \bar{g}(x, y, z) = (\zeta'.\eta').\xi'.$$

Hence $d(Rf, Rg) = d(\bar{f},\bar{g})$, by Theorem 5.4. But

$$d(Rf, Rg) = -\langle\alpha,\beta,\gamma\rangle,$$
$$d(\bar{g},\bar{f}) = (-1)^{rp+pq+qr}\langle -\gamma, -\beta, -\alpha\rangle,$$

and so Corollary 10.3 is obtained.

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THE PRODUCT OF BASIC BESSEL FUNCTIONS

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PUT

$$\prod_0^\infty (1-q^n xt)(1-q^n x^{-1}t) = \sum_{-\infty}^{\infty} (-1)^n x^n I_n(t), \tag{1}$$

$$\prod_0^\infty (1-q^n xt)^{-1}(1-q^n x^{-1}t)^{-1} = \sum_{-\infty}^{\infty} x^n \bar{I}_n(t), \tag{2}$$

where $|q| < 1$. The functions $I_n(t)$, $\bar{I}_n(t)$ were first introduced (in different notation) by Jackson (**4**, **5**) and have been discussed recently by Hahn (**3**). Since

$$I_n(t) = \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r(r-1)+\frac{1}{2}(n+r)(n+r-1)}t^{n+2r}}{(q)_r(q)_{n+r}}, \tag{3}$$

$$\bar{I}_n(t) = \sum_{r=0}^{\infty} \frac{t^{n+2r}}{(q)_r(q)_{n+r}}, \tag{4}$$

where $\qquad (q)_n = (1-q)(1-q^2)...(1-q^n), \qquad (q)_0 = 1$

and $\qquad \dfrac{1}{(q)_{-n}} = 0 \quad (n = 1, 2, 3,...),$

the functions may be called 'basic analogues' of the Bessel functions. Using (3) it is not difficult to verify that

$$\sum_{-\infty}^{\infty} q^{-\frac{1}{2}n(n-1)} x^n I_n(qt) = \prod_0^\infty \frac{1-q^{n+1}t^2}{(1-q^{n+1}xt)(1-q^n x^{-1}t)},$$

which implies $\qquad I_n(q^{\frac{1}{2}}t) = q^{\frac{1}{2}n^2}\bar{I}_n(t) \prod_{r=0}^{\infty} (1-q^r t^2). \qquad (5)$

In the next place put

$$\prod_0^\infty (1-q^n t)^{-1}(1-q^n xt)^{-1} = \sum_0^\infty H_n(x)t^n/(q)_n,$$

$$\prod_0^\infty (1-q^n t)(1-q^n xt) = \sum_0^\infty (-1)^n q^{\frac{1}{2}n(n-1)} G_n(x)t^n/(q)_n,$$

where

$$H_n(x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r, \qquad G_n(x) = \sum_{r=0}^{n} q^{r(r-n)} \begin{bmatrix} n \\ r \end{bmatrix} x^r,$$

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r(q)_{n-r}}, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1.$$

Orthogonality properties of $G_n(x)$ and $H_n(x)$ have been proved by Wigert (**7**) and Szegö (**6**), respectively. I have obtained (**1**, **2**) the bilinear generating relations

$$\sum_0^\infty \frac{t^n}{(q)_n} H_n(x)H_n(y) = \prod_0^\infty \frac{1-q^nxyt^2}{(1-q^nt)(1-q^nxt)(1-q^nyt)(1-q^nxyt)}, \tag{6}$$

$$\sum_0^\infty (-1)^n q^{\frac{1}{2}n(n+1)} \frac{t^n}{(q)_n} G_n(x)G_n(y)$$
$$= \prod_1^\infty \frac{(1-q^nt)(1-q^nxt)(1-q^nyt)(1-q^nxyt)}{1-q^nxyt^2}. \tag{7}$$

We now consider the sum

$$\sum_{n=-\infty}^{\infty} x^n \bar{I}_n(t)\bar{I}_n(z) = \sum_{n=-\infty}^{\infty} x^n \sum_r \frac{t^{n+2r}}{(q)_r(q)_{r+n}} \sum_s \frac{z^{n+2s}}{(q)_s(q)_{s+n}}$$

$$= \sum_{r,s,n} \frac{(xtz)^n t^{2r} z^{2s}}{(q)_r(q)_s(q)_{r+n}(q)_{s+n}} = \sum_{\substack{m,n,r,s\\ m-r=n-s}} \frac{(xtz)^{m-r} t^{2r} z^{2s}}{(q)_r(q)_s(q)_m(q)_n}$$

$$= \sum_{k=0}^\infty \frac{z^{2k}}{(q)_k^2} \sum_{m+s=k} \frac{(q)_k}{(q)_m(q)_s}\left(\frac{xt}{z}\right)^m \sum_{n+r=k} \frac{(q)_k}{(q)_r(q)_n}\left(\frac{t}{xz}\right)^r,$$

so that

$$\sum_{-\infty}^{\infty} x^n \bar{I}_n(t)\bar{I}_n(z) = \sum_0^\infty \frac{z^{2k}}{(q)_k^2} H_k\left(\frac{xt}{z}\right) H_k\left(\frac{t}{xz}\right). \tag{8}$$

The right-hand member of (8) differs from the left-hand member of (6) by a factor $(q)_k$. This suggests the use of the basic Laplace transform and its inverse as defined by Hahn [(**3**) 371]. If we let $\Phi(t)$ stand for the left-hand member of (6), then we have

$$\sum_{r=0}^\infty (-1)^r q^{\frac{1}{2}r(r+1)} \Phi(q^r t)/(q)_r = \sum_{n=0}^\infty \frac{t^n}{(q)_n} H_n(x)H_n(y) \sum_{r=0}^\infty (-1)^r q^{\frac{1}{2}r(r+1)} q^{rn}/(q)_r$$

$$= \sum_{n=0}^\infty \frac{t^n}{(q)_n} H_n(x)H_n(y) \prod_{r=1}^\infty (1-q^{n+r})$$

$$= Q \sum_{n=0}^\infty \frac{t^n}{(q)_n^2} H_n(x)H_n(y),$$

where

$$Q = \prod_{n=1}^\infty (1-q^n).$$

If we now replace t, x, y by z^2, xt/z, t/xz, respectively, we get

$$Q\sum_{n=0}^{\infty}\frac{z^{2n}}{(q)_n^2}H_n\left(\frac{xt}{z}\right)H_n\left(\frac{t}{xz}\right)$$

$$=\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+1)}\prod_{n=0}^{\infty}\frac{1-q^{n+2r}t^2z^2}{(1-q^{n+r}t^2)(1-q^{n+r}z^2)(1-q^{n+r}xtz)(1-q^{n+r}x^{-1}tz)}$$

$$=\prod_{s=0}^{\infty}\frac{1-q^st^2z^2}{(1-q^st^2)(1-q^sz^2)}\times$$

$$\times\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+1)}\frac{(t^2)_r(z^2)_r}{(t^2z^2)_{2r}}\prod_{n=0}^{\infty}(1-q^{n+r}xtz)^{-1}(1-q^{n+r}x^{-1}tz)^{-1}$$

$$=\frac{e(t^2)e(z^2)}{e(t^2z^2)}\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+1)}\frac{(t^2)_r(z^2)_r}{(t^2z^2)_{2r}}\sum_{n=-\infty}^{\infty}x^n\bar{I}_n(q^rtz),$$

where

$$e(t)=\prod_0^{\infty}(1-q^nt)^{-1}, \tag{9}$$

$$(t)_r=(1-t)(1-qt)\dots(1-q^{r-1}t). \tag{10}$$

Thus, comparing with (8), we get

$$Q\sum_{-\infty}^{\infty}x^n\bar{I}_n(t)\bar{I}_n(z)=\frac{e(t^2)e(z^2)}{e(t^2z^2)}\sum_{n=-\infty}^{\infty}x^n\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+1)}\frac{(t^2)_r(z^2)_r}{(t^2z^2)_{2r}}\bar{I}_n(q^rtz). \tag{11}$$

Equating coefficients, we obtain

$$Q\bar{I}_n(t)\bar{I}_n(z)=\frac{e(t^2)e(z^2)}{e(t^2z^2)}\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+1)}\frac{(t^2)_r(z^2)_r}{(t^2z^2)_{2r}}\bar{I}_n(q^rtz). \tag{12}$$

To get the inverse of (12), we return to (8) and replace t, z by $tz^{\frac{1}{2}}$, $z^{\frac{1}{2}}$, respectively, so that

$$\sum_{-\infty}^{\infty}x^n\bar{I}_n(tz^{\frac{1}{2}})\bar{I}_n(z^{\frac{1}{2}})=\sum_0^{\infty}\frac{z^k}{(q)_k^2}H_k(xt)H_k(x^{-1}t).$$

It follows that

$$\sum_{n=-\infty}^{\infty}x^n\sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\bar{I}_n(q^{\frac{1}{2}r}tz^{\frac{1}{2}})\bar{I}_n(q^{\frac{1}{2}r}z^{\frac{1}{2}})$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{(q)_k^2}H_k(xt)H_k(x^{-1}t)\sum_{r=0}^{\infty}\frac{q^{rk+r}}{(q)_r}=Q^{-1}\sum_{k=0}^{\infty}\frac{z^k}{(q)_k}H_k(xt)H_k(x^{-1}t)$$

$$=Q^{-1}\prod_{n=0}^{\infty}\frac{1-q^nt^2z^2}{(1-q^nz)(1-qt^2z)(1-q^nxtz)(1-q^nx^{-1}tz)}$$

$$=Q^{-1}\frac{e(z)e(t^2z)}{e(t^2z^2)}\sum_{-\infty}^{\infty}x^n\bar{I}_n(tz).$$

Consequently

$$Q^{-1}\frac{e(z)e(t^2z)}{e(t^2z^2)}\bar{I}_n(tz) = \sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\bar{I}_n(q^{\frac12 r}tz^{\frac12})\bar{I}_n(q^{\frac12 r}z^{\frac12}).$$

Finally, replacing $tz^{\frac12}$, $z^{\frac12}$ by t and z, respectively, we obtain

$$Q^{-1}\frac{e(t^2)e(z^2)}{e(t^2z^2)}\bar{I}_n(tz) = \sum_{r=0}^{\infty}\frac{q^r}{(q)_r}\bar{I}_n(q^{\frac12 r}t)\bar{I}_n(q^{\frac12 r}z). \tag{13}$$

To obtain the corresponding formulae for I_n, we substitute from (5) in (12) and (13). We find that

$$QI_n(q^{\frac12}t)I_n(q^{\frac12}z) = q^{\frac12 n^2}\sum_{r=0}^{\infty}(-1)^r\frac{q^{\frac12 r(r+1)}(t^2)_r(z^2)_r}{(q)_r(t^2z^2)_r}I_n(q^{r+\frac12}tz), \tag{14}$$

$$Q^{-1}q^{\frac12 n^2}I_n(q^{-\frac12}tz) = \sum_{r=0}^{\infty}\frac{q^r}{(q)_r}(q^{-1}t^2)_r(q^{-1}z^2)_r\,I_n(q^{\frac12 r}t)I_n(q^{\frac12 r}z). \tag{15}$$

If we apply the method used in proving (12) and (13) directly to (7) and (3), we do not obtain (14) and (15). However, we get some other identities that may be of interest. Thus from (3) it follows that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x^n I_n(t)I_n(z)
&= \sum_n q^{n(n-1)}x^n \sum_r \frac{q^{r^2-r+rn}t^{n+2r}}{(q)_r(q)_{n+r}}\sum_s \frac{q^{s^2-s+sn}z^{n+2s}}{(q)_s(q)_{n+s}}\\
&= \sum_{r,s,n} q^{n(n-1)}\frac{q^{r^2+s^2+(r+s)(n-1)}(xtz)^n t^{2r}z^{2s}}{(q)_r(q)_s(q)_{r+n}(q)_{s+n}}\\
&= \sum_{\substack{r,s,m,n\\ m-r=n-s}} q^{(m-r)(m-r-1)}\frac{q^{r^2+s^2+(r+s)(m-r-1)}(xtz)^{m-r}t^{2r}z^{2s}}{(q)_r(q)_s(q)_m(q)_n}\\
&= \sum_{k=0}^{\infty} q^{k(k-1)}\frac{z^{2k}}{(q)_k^2}\sum_{m+s=k} q^{m(m-k)}\begin{bmatrix}k\\m\end{bmatrix}\left(\frac{xt}{z}\right)^m \sum_{n+r=k} q^{r(r-k)}\begin{bmatrix}k\\r\end{bmatrix}\left(\frac{t}{xz}\right)^r
\end{aligned}$$

since

$$(m-r)(m-r-1)+r^2+(k-m)^2+(r+k-m)(m-r-1)$$
$$= k(k-1)+m(m-k)+r(r-k).$$

We have therefore

$$\sum_{-\infty}^{\infty} x^n I_n(t)I_n(z) = \sum_{0}^{\infty} q^{k(k-1)}\frac{z^{2k}}{(q)_k^2}G_k\left(\frac{xt}{z}\right)G_k\left(\frac{t}{xz}\right). \tag{16}$$

Since (16) implies

$$\sum_{-\infty}^{\infty} x^n I_n(tz^{\frac12})I_n(z^{\frac12}) = \sum_{0}^{\infty} q^{k(k-1)}\frac{z^k}{(q)_k^2}G_k(xt)G_k(x^{-1}t),$$

it follows, exactly as before, that

$$\sum_{n=-\infty}^{\infty} x^n \sum_{r=0}^{\infty} \frac{q^r}{(q)_r} I_n(q^{\frac{1}{2}r}tz^{\frac{1}{2}}) I_n(q^{\frac{1}{2}r}z^{\frac{1}{2}}) = Q^{-1} \sum_{k=0}^{\infty} q^{k(k-1)} \frac{z^k}{(q)_k} G_k(xt) G_k(x^{-1}t),$$

or, what is the same thing,

$$\sum_{n=-\infty}^{\infty} x^n \sum_{r=0}^{\infty} \frac{q^r}{(q)_r} I_n(q^{\frac{1}{2}r}t) I_n(q^{\frac{1}{2}r}z) = Q^{-1} \sum_{k=0}^{\infty} q^{k(k-1)} \frac{z^{2k}}{(q)_k} G_k\left(\frac{xt}{z}\right) G_k\left(\frac{t}{xz}\right). \tag{17}$$

However, (7) cannot be applied to the right-hand side of (17). On the other hand, from (7) we get

$$Q^{-1} \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{t^n}{(q)_n^2} G_n(x) G_n(y) = \sum_{r=0}^{\infty} \frac{q^r e(q^{2r-1}xyt^2)}{(q)_r e(q^r t) e(q^r xt) e(q^r yt) e(q^r xyt)}.$$

Replacing t, x, y by z^2, xt/z, t/xz, respectively, we get for the right-hand member

$$\sum_{r=0}^{\infty} \frac{q^r e(q^{2r-1}t^2z^2)}{(q)_r e(q^r t^2) e(q^r z^2) e(q^r xtz) e(q^r x^{-1}tz)}$$
$$= \frac{e(q^{-1}t^2z^2)}{e(t^2)e(z^2)} \sum_{r=0}^{\infty} \frac{q^r (q^{-1}t^2z^2)_{2r}}{(q)_r (t^2)_r (z^2)_r} \sum_{n=-\infty}^{\infty} x^n I_n(q^r tz).$$

Thus we get

$$Q^{-1} \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{z^{2n}}{(q)_n^2} G_n\left(\frac{xt}{z}\right) G_n\left(\frac{t}{xz}\right)$$
$$= \frac{e(q^{-1}t^2z^2)}{e(t^2)e(z^2)} \sum_{n=-\infty}^{\infty} x^n \sum_{r=0}^{\infty} \frac{q^r (q^{-1}t^2z^2)_{2r}}{(q)_r (t^2)_r (z^2)_r} I_n(q^r tz). \tag{18}$$

Returning to (16), if we substitute from (5), we get

$$\sum_{-\infty}^{\infty} q^{n^2} x^n \bar{I}_n(t) \bar{I}_n(z) = e(t^2) e(z^2) \sum_{0}^{\infty} q^{k^2} \frac{z^{2k}}{(q_k)^2} G_k\left(\frac{xt}{z}\right) G_k\left(\frac{t}{xz}\right), \tag{19}$$

which may be compared with (8).

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MUTABILITY OF BIFURCATING ROOT-TREES

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Synopsis. If α, δ, μ are the altitude, potency, and mutability of a bifurcating root-tree, then $\alpha-\mu \leqslant \log_2(\delta-\mu)$ and $\mu \geqslant h-1$, where h is the sum of digits in δ written in the binary scale of notation. It is shown constructively that, for given non-negative integers α, δ satisfying $\alpha+1 \leqslant \delta \leqslant 2^\alpha$, the least integer μ which satisfies the above relations is the mutability of a bifurcating root-tree.

Bifurcating root-trees can be defined as follows (**3**):

(i) a single knot $\cdot$ is a bifurcating root-tree, the knot being the root of the tree;

(ii) if (P) and (Q) are bifurcating root-trees, then ${}^{(P)}\mathbf{V}^{(Q)}$, where the free (upper) knots of the fork $\mathbf{V}$ are connected to the roots of (P) and (Q), is a bifurcating root-tree.

Trees will mean bifurcating root-trees.

Indices of the free logarithmetic are elements of the free additive groupoid generated by element 1. There is a one–one correspondence between indices and trees: $1 \leftrightarrow \cdot$ and, if $P \leftrightarrow (P)$ and $Q \leftrightarrow (Q)$, then $P+Q \leftrightarrow {}^{(P)}\mathbf{V}^{(Q)}$. We can therefore use the terminologies and the notations for indices and trees indiscriminately. It is convenient to denote $1+1$ by 2 and $2^{i-1}+2^{i-1}$ by 2^i. The index 2^i then corresponds to a tree with all its free knots at the maximal altitude (**1**).

Altitude α_P, *potency* δ_P (called 'degree' by Etherington), and ***mutability*** μ_P of a tree P can be defined as follows:

$$\alpha_1 = 0, \quad \alpha_{P+Q} = 1+\max(\alpha_P, \alpha_Q);$$

$$\delta_1 = 1, \quad \delta_{P+Q} = \delta_P+\delta_Q;$$

$$\mu_1 = 0, \quad \mu_{P+Q} = \begin{cases} 2\mu_P & \text{if } P,\, Q \text{ are } \textit{conformal} \text{ (1), i.e. if the corresponding indices become equal when addition of indices is commutative,} \\ \mu_P+\mu_Q+1 & \text{if } P,\, Q \text{ are not conformal.} \end{cases}$$

The altitude of a tree is the distance (the number of forks) from its root to its furthest free knot; the potency is the number of free knots in the tree. If $P = R+S$, then R and S are called *subtrees* of first order

of P. If T is a subtree of first order of a subtree of $(n-1)$th order of P, then T is a subtree of nth order of P. A knot of P is called *unbalanced* (**1**) if the subtree of P of which the knot is the root has non-conformal subtrees of first order; otherwise it is *balanced*. The mutability of P is equal to the number of unbalanced knots in P. Thus the number of trees conformal to a given tree of mutability μ is 2^{μ}.

Etherington has shown (**1**) that α, δ, μ must satisfy the conditions

(i) $2^{\alpha} \geqslant \delta \geqslant \alpha+1$;

(ii) $\delta \geqslant \mu+2$ $(\delta \neq 1)$,

the equality holding if and only if $\delta = \alpha+1$;

(iii) $\mu \leqslant 3.2^{\alpha-3}-1$ $(\alpha \geqslant 3)$.

For a given δ or a given α conditions (ii), (iii) prescribe the maximal value for μ. In the present paper I find two minimal conditions for μ and show that for any given non-negative integers α, δ satisfying condition (i) the least number satisfying these minimal conditions is in fact the mutability of a tree of altitude α and potency δ.

Potency δ_P can be expressed uniquely as a sum of h_P distinct powers of 2:

$$\delta = 2^{i_1}+2^{i_2}+\ldots+2^{i_h} \quad (i_1 > i_2 > \ldots > i_h \geqslant 0).$$

Obviously $i_1 = [\log_2 \delta]$ and h is equal to the sum of digits in δ written in the binary scale of notation. We have the lemma:

Lemma. $h_P+h_Q \geqslant h_{P+Q}$.

For, clearly, in any scale of notation the total sum of digits in several natural numbers cannot be exceeded by that of the sum of these numbers. We deduce the theorems:

Theorem 1. *Let δ and μ be the potency and the mutability of a tree P and let $\delta = 2^{i_1}+2^{i_2}+\ldots+2^{i_h}$ $(i_1 > i_2 > \ldots > i_h)$. Then $\mu \geqslant h-1$.*

Proof. Use non-associative induction [cf. (**2**), or (**3**) 322]. If $P = 1$, $\mu = 0$, $\delta = 2^0$, and hence $h = 1$ and $\mu = h-1$. Let $P = Q+R$ and assume that the theorem holds for Q and R. Then (i), if Q and R are not conformal,

$$\begin{aligned}\mu_{Q+R} &= \mu_Q+\mu_R+1\\ &\geqslant (h_Q-1)+(h_R-1)+1, \quad \text{by the induction hypothesis,}\\ &= (h_Q+h_R)-1\\ &\geqslant h_{Q+R}-1, \quad \text{by the preceding lemma;}\end{aligned}$$

(ii), if Q and R are conformal,

$$\begin{aligned}\mu_{Q+R} &= 2\mu_Q \\ &\geqslant 2h_Q-2, \quad \text{by the induction hypothesis,} \\ &= 2h_{Q+R}-2 \\ &\geqslant h_{Q+R}-1, \quad \text{since } h_{Q+R} \geqslant 1.\end{aligned}$$

THEOREM 2. *If α, δ, μ are the altitude, potency, and mutability of a tree P, then $\alpha-\mu \leqslant \log_2(\delta-\mu)$.*

Proof. If $P = 1$, then

$$\alpha-\mu = 0-0 = \log_2(1-0) = \log_2(\delta-\mu).$$

Let $P = Q+R$ and assume that

$$\alpha_Q-\mu_Q \leqslant \log_2(\delta_Q-\mu_Q), \qquad \alpha_R-\mu_R \leqslant \log_2(\delta_R-\mu_R).$$

Without loss of generality assume that $\alpha_Q \geqslant \alpha_R$. Then

(i) if Q, R are not conformal,

$$\begin{aligned}\alpha_{Q+R}-\mu_{Q+R} &= (\alpha_Q+1)-(\mu_Q+\mu_R+1) \\ &= (\alpha_Q-\mu_Q)+(\alpha_R-\mu_R)-\alpha_R \\ &\leqslant \log_2(\delta_Q-\mu_Q)+\log_2(\delta_R-\mu_R)-\log_2\delta_R, \\ &\qquad \text{from the induction hypothesis and } \alpha \geqslant \log_2\delta, \\ &= \log_2\{\delta_Q-\mu_Q+\mu_R\delta_R^{-1}(\mu_Q-\delta_Q)\} \\ &\leqslant \log_2(\delta_Q-\mu_Q+\delta_R-\mu_R-1), \\ &\qquad \text{since } \mu_R\delta_R^{-1}(\mu_Q-\delta_Q) \leqslant 0 \text{ while } \delta_R-\mu_R-1 \geqslant 0 \\ &= \log_2(\delta_{Q+R}-\mu_{Q+R}).\end{aligned}$$

(ii), if Q, R are conformal, then $\delta_Q = \delta_R$, $\mu_Q = \mu_R$ and

$$\begin{aligned}\alpha_{Q+R}-\mu_{Q+R} &= (\alpha_Q+1)-2\mu_Q \\ &= (\alpha_Q-\mu_Q)+\log_2 2-\mu_Q \\ &\leqslant \log_2(2\delta_Q-2\mu_Q)-\mu_Q, \quad \text{by the induction hypothesis,} \\ &\leqslant \log_2(2\delta_Q-2\mu_Q) \\ &= \log_2(\delta_{Q+R}-\mu_{Q+R}).\end{aligned}$$

THEOREM 3. *Let α, δ be non-negative integers such that $\alpha+1 \leqslant \delta \leqslant 2^\alpha$ and let*

$$\delta = 2^{i_1}+2^{i_2}+\ldots+2^{i_h} \quad (i_1 > i_2 > \ldots > i_h \geqslant 0).$$

If μ is the least integer such that

$$\alpha-\mu \leqslant \log_2(\delta-\mu) \quad \textit{and} \quad \mu \geqslant h-1,$$

then there exists a tree (in general not unique) of altitude α, potency δ, and mutability μ.

Proof. (i) If $\alpha \leqslant i_1+h-1$ and $h \neq 1$, then the tree

$$\llbracket[\{(2^{i_1}+2^{i_2})+2^{i_3}\}+\ldots]+2^{i_p}\rrbracket+\llbracket 2^{i_{p+1}}+[2^{i_{p+2}}+\{\ldots+(2^{i_{h-1}}+2^{i_h})\}]\rrbracket,$$

where $p = \alpha-i_1 < h$, has altitude α, potency δ, and mutability $h-1$. By Theorem 2,

$$\alpha-(h-1) \leqslant \log_2\{\delta-(h-1)\}.$$

Hence, by definition of μ, $\mu = h-1$.

If $h = 1$ and $\alpha \leqslant i_1+h-1$, then $\alpha \leqslant i_1 = \log_2\delta$ and, since $\delta \leqslant 2^\alpha$, $\delta = 2^\alpha$ and the only tree of altitude α and potency δ is the tree 2^α, a tree of mutability $0 = h-1$.

(ii) If $\alpha \geqslant i_1+h$, we can show that δ can be expressed in the form

$$\delta = 2^{j_1}+2^{j_2}+\ldots+2^{j_k},$$

where k, the number of terms, is equal to $\alpha-j_1+1$ and either (*a*)

$$j_1 > j_2 > \ldots > j_{k-1} = j_k > 0 \quad (3 \leqslant k \leqslant j_1+1),$$

or (*b*)

$$j_1 > j_2 > \ldots > j_{r-1} = j_r > j_{r+1} = j_{r+2} = \ldots = j_k = 0$$
$$(3 \leqslant r \leqslant j_1+1,\ r \leqslant k-1),$$

or (*c*)

$$j_1 > j_2 > \ldots > j_s > j_{s+1} = \ldots = j_k = 0$$
$$(1 \leqslant s \leqslant j_1,\ s \leqslant k-2,\ 3 \leqslant k \leqslant \delta-1).$$

The proof is by induction on α. Note that $\alpha \geqslant i_1+h$ implies $\alpha \geqslant 3$ and that $\alpha = 3$ together with $\alpha \geqslant i_1+h$ and $\alpha+1 \leqslant \delta \leqslant 2^\alpha$ imply $\delta = 4$. Thus, if $\alpha = 3$, we can express δ in the form $2+1+1$ satisfying (*c*). Now let $\alpha > 3$ and assume that δ can be expressed in the form

$$\delta = 2^{j_1}+2^{j_2}+\ldots+2^{j_m}, \tag{1}$$

where m, the number of terms, is equal to $\alpha-j_1$ and the j's satisfy (*a*), (*b*), or (*c*). I show that we can obtain from (1) an expansion for δ in which the number of terms plus the greatest exponent is equal to $\alpha+1$

and the exponents satisfy (a), (b), or (c). If the j's in (1) satisfy (a), the required expansion is

$$\delta = 2^{j_1}+2^{j_2}+...+2^{j_m-1}+2^{j_m-1};$$

if they satisfy (b) the required expansion is

$$\delta = 2^{j_1}+2^{j_2}+...+2^{j_{r-1}}+2^{j_r-1}+2^{j_r-1}+2^{j_{r+1}}+...+2^{j_m};$$

if they satisfy (c) and $s \neq 1$, the required expansion is

$$\delta = 2^{j_1}+2^{j_2}+...+2^{j_{s-1}}+2^{j_s-1}+2^{j_s-1}+2^{j_{s+1}}+...+2^{j_m};$$

if $s = 1$, it is

$$\delta = 2^{j_1-1}+2^{j_1-2}+2^{j_1-2}+2^{j_2}+...+2^{j_m}.$$

Note that in (1) $j_1 \geqslant 2$, and thus the last expansion is possible, for $j_1 = 1$ implies that the number of terms is $\delta-1$, which cannot be equal to $\alpha-j_1 = \alpha-1$.

Let then $$\delta = 2^{j_1}+2^{j_2}+...+2^{j_k},$$

where $k = \alpha-j_1+1$ and the j's satisfy (a), (b), or (c), and consider the tree $[\{(2^{j_1}+2^{j_2})+2^{j_3}\}+...]+2^{j_k}$. Its altitude is α, its potency δ, and mutability $k-1$. Since α, δ, $k-1$ are altitude, potency, and mutability of a tree, it follows from Theorems 1 and 2 that

$$\alpha-(k-1) \leqslant \log_2\{\delta-(k-1)\}, \qquad k-1 \geqslant h-1.$$

Let $t > 1$. We have to show that at least one of the two relations

$$\alpha-(k-t) \leqslant \log_2\{\delta-(k-t)\}, \qquad k-t \geqslant h-1$$

is false. I shall show that in fact the first of these is false.

If the expansion $\delta = 2^{j_1}+2^{j_2}+...+2^{j_k}$ is of the form (a), then

$$\delta \leqslant 2^{j_1}+2^{j_1-1}+...+2^{j_1-(k-2)}+2^{j_1-(k-2)}$$
$$= 2^{j_1+1} \leqslant 2^{j_1+1}+k-3, \quad \text{since } k \geqslant 3;$$

if it is of the form (b),

$$\delta \leqslant 2^{j_1}+2^{j_1-1}+...+2^{j_1-(r-2)}+2^{j_1-(r-2)}+(k-r)$$
$$= 2^{j_1+1}+k-r \leqslant 2^{j_1+1}+k-3, \quad \text{since } r \geqslant 3;$$

if it is of the form (c),

$$\delta \leqslant 2^{j_1}+2^{j_1-1}+...+2^{j_1-(s-2)}+2^{j_1-(s-1)}+(k-s)$$
$$= 2^{j_1+1}-2^{j_1-(s-1)}+k-s \leqslant 2^{j_1+1}+k-3,$$

since $1 \leqslant s \leqslant j_1$ and therefore $2^{j_1-(s-1)}+s \geqslant 3$. Thus in all cases

$$\delta \leqslant 2^{j_1+1}+k-3.$$

Hence

$$\begin{aligned}
\alpha-(k-t) &= j_1+t-1 \\
&= \log_2 2^{j_1+t-1} \\
&= \log_2\{2^{j_1+1}.2^{t-2}\} \\
&\geqslant \log_2\{(\delta-k+3).2^{t-2}\} \\
&\geqslant \log_2\{(\delta-k+3)+(t-2)\} \\
&> \log_2\{\delta-(k-t)\}.
\end{aligned}$$

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ON GROUPS OVER A D.G. NEAR-RING (I): SUM CONSTRUCTIONS AND FREE R-GROUPS

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Introduction

IN the present and subsequent papers the notion of a group over a distributively generated near-ring R [cf. (**1**), (**2**)], or briefly of an 'R-group'† is to be developed as a unifying concept for both modules over a ring and groups (not necessarily abelian). The aim of these papers is to extend some of the basic ideas and tools of module theory to the non-abelian case. By taking R to be the ring of integers Z, R-groups specialize to groups in the ordinary sense. On the other hand, by taking the underlying group-structure to be commutative we obtain a specialization to modules over a ring; in fact an abelian R-group is a module over the residue ring of R *modulo* its commutator ideal [cf. (**1**) § 4]. As will be shown elsewhere the ideas and techniques of homological algebra can be applied to R-groups. The forthcoming series of papers thus provides the basis for the non-abelian generalization of homology and cohomology theory of modules.

The 'operator'-domains R one wishes to consider are those generated by group-endomorphisms with the 'natural' addition and multiplication of mappings. One thus arrives at systems with two binary operations. The first problem is then to characterize such systems axiomatically. If the underlying groups are commutative, this leads to the abstract concept of a ring; in (**1**) it was shown that the abstract counterpart of these domains in the general case is the d.g. near-ring.‡ The group over such a near-ring appears thus as the natural generalization of the module over a ring.

In the module case every element of the ring induces an endomorphism. This no longer extends to R-groups. To each R-group there

† The term 'R-group' is also used with another meaning in the theory of groups with root extraction; no confusion should be possible, however, as the context is quite different. It seems preferable to use this term and to retain the alternative 'R-module' in its accepted meaning, as referring to abelian groups with operator ring R.

‡ 'd.g.' is used as a standard abbreviation for 'distributively generated'.

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corresponds, however, a subset of elements in R which induce endomorphisms. These elements form a semi-group S in R under multiplication, satisfying further conditions;† S may be taken as a distributive semigroup generating R [cf. (**1**) § 1]. It is then natural to use the notion of an (R, S)-group, i.e. of an R-group which admits S as a domain of endomorphisms. Every R-group is, of course, an (R, S)-group for some S. In particular every group is a $(Z, 1)$-group.‡ If R is a ring, then every R-module is an $(R, R^{\times})$-group,§ and the converse is equally true, provided that R has an identity inducing the identity mapping. We shall see that the natural categories of R-groups to be studied are categories of (R, S)-groups for fixed S.

In § 1 the basic concepts are introduced. In § 2 we construct the free near-ring $Z(S)$ of a multiplicative semigroup S. These near-rings, having a relatively transparent structure, form an important tool in the sequel. They seem also to be of significance in other respects, e.g. for the theory of representation of a semigroup; if S is a group, $Z(S)$ is the obvious non-abelian generalization of the group-ring over the integers.

In § 3 and § 4 we study processes of constructing new R-groups from given ones; § 3 contains an axiomatic treatment of free sums‖ and cartesian sums†† which exhibits their duality and may even be of some interest in the known case $R = Z$. The cartesian sum of R-groups is a well-known group construction endowed with a natural R-group structure. The underlying abstract group of a 'free (R, S)-sum' is, however, not in general the free sum, or, if multiplicative notation is used, the free product, of the component groups. In fact, the specialization to modules over a ring leads to the direct sum of modules (cf. § 3, Example 1). In § 4 we discuss orthogonal sums and obtain further characterizations of the new constructions, and also of the direct sum. The results of these two sections are of some importance for the study of certain functors to be defined in subsequent papers.

In the concluding § 5 we introduce free (R, S)-groups, which are to play an essential role in the further theory.

† *Modulo* the annihilating ideal in R of the given R-group, S will be (left) distributive and will generate the additive group of R.

‡ Z is throughout the ring of integers and the symbol 1 stands both for the identity of Z and for the semigroup consisting of the identity.

§ $R^{\times}$ is the multiplicative semigroup of R.

‖ 'Free product' in multiplicative terminology.

†† 'Cartesian product' or 'complete direct product' in multiplicative terminology.

1. Basic concepts

Unless otherwise mentioned, additive notation will be used for groups: this is not to imply commutativity. We recall the definition of a d.g. ('distributively generated') near-ring [cf. (**1**) 1] as an algebraic system with two binary operations, addition and multiplication, such that

A (i) *the elements of R form a group R^+ under addition*;

(ii) *multiplication is associative, i.e. the elements of R form a multiplicative semigroup $R^\times$*;

(iii) *for all x, y, z of R, $(x+y)z = xz+yz$*;

(iv) *the set of elements t with $t(x+y) = tx+ty$, for all x, y of R, is a set of generators of R^+.*

Postulate (iv) may be replaced by

(iv′) *there exists a multiplicative semigroup S in $R^\times$ whose elements generate R^+, and such that, for all s of S and all x, y of R,*

$$s(x+y) = sx+sy.$$

Such a semigroup will be called a *distributive semigroup generating R*.

An R-group structure μ on a group Ω is given by a mapping μ: $(x,\omega) \to x\omega$ of $R \times \Omega$ into Ω such that

B (i) $(x+y)\omega = x\omega+y\omega$, *for all x, y of R and all ω of Ω*;

(ii) $(xy)\omega = x(y\omega)$, *for all x, y of R and all ω of Ω*;

(iii) *there exists a distributive semigroup S in R generating R such that, for all s of S and all ω_1, ω_2 of Ω, $s(\omega_1+\omega_2) = s\omega_1+s\omega_2$.*

A group with given R-group structure is an R-group; if we wish to specify the semigroup S in B (iii) explicitly we shall speak of an '(R, S)-group'.

Next let S be a multiplicative semigroup not necessarily embedded in a near-ring. An S-group structure ν on a group Ω is given by a mapping ν: $(s,\omega) \to s\omega$ of $S \times \Omega$ into Ω such that

C (i) $(st)\omega = s(t\omega)$, *for all s, t of S and all ω of Ω*;

(ii) $s(\omega_1+\omega_2) = s\omega_1+s\omega_2$, *for all s of S and all ω_1, ω_2 of Ω.*

A group with given S-group structure is an S-group.

Let now S be a distributive semigroup generating the near-ring R. If μ is an R-group structure making Ω into an (R, S)-group, then the restriction ν of μ to $S \times \Omega$ defines an S-group structure on Ω.

Conversely, if the restriction ν of μ to $S\times\Omega$ defines an S-group structure, then Ω is an (R, S)-group, and μ is determined by ν.

If f, g are mappings of a group Ω into a group Δ,† we define their sum $f+g$ by

$$(f+g)\omega = f\omega+g\omega \quad \text{for all } \omega \text{ of } \Omega. \tag{1.1}$$

(*Natural mapping addition.*) If $f\colon \Omega\to\Delta$, $g\colon \Delta\to\Gamma$ are group mappings, we define their product $g\circ f$ by composition:

$$(g\circ f)\omega = g(f\omega) \quad \text{for all } \omega \text{ of } \Omega. \tag{1.2}$$

(*Mapping multiplication.*) A semigroup S whose elements are endomorphisms of a group Ω and whose operation is mapping multiplication is called a 'semigroup of endomorphisms of Ω'. This amounts to saying that Ω is an S-group and that, if for s_1, s_2 of S and for all ω of Ω, $s_1\omega = s_2\omega$, then $s_1 = s_2$. A near-ring R whose elements are mappings $\Omega\to\Omega$ and whose operations are mapping addition and mapping multiplication is called a 'near-ring of mappings of Ω'. Particular examples are the near-ring $R(\Omega)$ of all mappings $\Omega\to\Omega$, and the near-ring of mappings $\Omega\to\Omega$ generated by a semigroup S of endomorphisms of Ω. This latter near-ring is generated by the distributive semigroup S and may be considered as a sub-near-ring of $R(\Omega)$.

We note that the postulates B (i)–(iii) are stronger than those used in (**2**). It is, however, easily verified that again all R-subgroups and all R-homomorphic images of an (R, S)-group are (R, S)-groups according to our definition. In particular all groups are $(Z, 1)$-groups, where Z is the ring of integers. The term 'Z-group' will always be used in this natural sense, i.e. we shall always assume that $1\omega = \omega$. Moreover, if R_1 is a near-ring of mappings of Ω generated by the semigroup S_1 of endomorphisms of Ω, then Ω is an (R_1, S_1)-group. The representations of a d.g. near-ring R by such near-rings R_1 then correspond to the R-groups Ω.

The R-subgroups of R^+ are, also under the postulates given here, precisely the left R-modules of R [cf. (**1**) § 2] and the R-quotient groups of R^+ are the quotient groups of R^+ *modulo* left ideals in R.

It has become useful in various contexts to express the notion of a representation of an algebraic system in terms of its extension by ideals. On the basis of our system of postulates for an R-group this approach is natural also for d.g. near-rings. The problem is that of constructing d.g. extension near-rings R' of R such that (i) the additive group of R' is the direct sum of R^+ and of a group Ω, (ii) Ω is an ideal

† Mappings are written on the left.

in R' annihilated on the right by R'. The solutions to this extension problem are given precisely by the R-groups Ω in the sense of B.

2. The free near-ring $Z(S)$

We consider a multiplicative semigroup S with $s_1 s_2$ denoting the product of s_1, s_2 in S. Let $\Sigma(S) = \Sigma$ be the free group on the symbols $[u]$, and $[su]$ for all s of S. For each t of S there exists a unique endomorphism of Σ, again denoted by t, such that

$$t[u] = [tu], \quad t[su] = [(ts)u] \quad \text{for all } s \text{ of } S.$$

We verify that for t_1, t_2 of S the endomorphism $t_1 t_2$ and the mapping product $t_1 \circ t_2$ of the endomorphisms t_1 and t_2 coincide. It suffices of course to show that $t_1(t_2 \sigma) = (t_1 t_2)\sigma$ for all free generators σ of Σ. Thus S is a semigroup of endomorphims of Σ. We denote the near-ring of mappings of Σ which is generated by S, by the symbol $Z(S)$. $Z(S)$ is the *free near-ring* of S.

THEOREM 2.1. (i) *$Z(S)$ is a d.g. near-ring generated by the distributive semigroup S;*

(ii) *$Z(S)^+$ is the free group on the set S;*

(iii) *every S-group Ω is a $Z(S)$-group;*†

(iv) *every homomorphism θ of S into the multiplicative semigroup $R^\times$ of a near-ring R, such that θS is a distributive semigroup generating R, can be extended to a near-ring epimorphism $\bar{\theta}$: $Z(S) \to R$.*

$Z(S)$ is uniquely determined to within near-ring isomorphism by (i) *and either* (ii), *or* (iii), *or* (iv).

Proof.‡ (i) follows from the definition [cf. (**1**) § 1, Example 3].

For (ii) we consider the subgroup $\Sigma'(S) = \Sigma'$ generated by the symbols $[su]$ for all s of S. Σ' is then the free group on these symbols. Every generator of this group is of the form $x[u]$ ($x \in Z(S)$); using the distributive law B (i) we conclude that all elements of Σ' are of this form. Conversely, since S generates $Z(S)^+$, we have $x[u] \in \Sigma'$ for all x of $Z(S)$. To establish (ii) it thus suffices to show that the mapping $x \to x[u]$ is an isomorphism of groups. By B (i) and by what we have

† More precisely the S-group structure extends to a $Z(S)$-group structure.

‡ It is essential in the following proof to distinguish symbolically some of the several 'multiplicative' operations which occur. In accordance with its original derivation the product in $Z(S)$ is denoted by $x \circ y$. We shall also have to introduce a multiplication $x.\omega$ of elements ω of an S-group Ω by elements x of $Z(S)$. All other products will in the usual way be denoted by juxtaposition.

just seen, the mapping is an epimorphism. We have to show that $x[u] = 0$ implies $x\sigma = 0$ for all σ of Σ.

If $\sigma \in \Sigma$, there exists a unique endomorphism ϕ of Σ with $\phi[u] = \sigma$ and $\phi[su] = s\sigma$, for all s of S. If $t \in S$, we verify that

$$(\phi \circ t)[u] = (t \circ \phi)[u], \qquad (\phi \circ t)[su] = (t \circ \phi)[su],$$

for all s of S. The endomorphisms $\phi \circ t$ and $t \circ \phi$ coincide on the generators. Hence ϕ is an S-endomorphism, and so a $Z(S)$-endomorphism [cf. (**2**), 2.1.1]. In particular

$$x\sigma = x(\phi[u]) = \phi(x[u]) = 0$$

if $x[u] = 0$.

Let Ω be an S-group. Using (ii) we see that for each ω of Ω there exists a unique homomorphism $Z(S)^+ \to \Omega$ of additive groups such that, if we denote the image of x by $x.\omega$, we have

$$s.\omega = s\omega \quad \text{for all } s \text{ of } S. \tag{2.1}$$

Since $x \to x.\omega$ is a homomorphism, we have

$$(x+y).\omega = x.\omega + y.\omega \quad \text{for all } x, y \text{ of } Z(S). \tag{2.2}$$

It remains to establish that

$$(x \circ y).\omega = x.(y.\omega). \tag{2.3}$$

Let S be an element of S. Using (2.1), which gives

$$s.(\omega_1 + \omega_2) = s.\omega_1 + s.\omega_2,$$

and (2.2), we deduce that

$$s.((x+y).\omega) = s.(x.\omega) + s.(y.\omega),$$

$$(s \circ (x+y)).\omega = (s \circ x).\omega + (s \circ y).\omega,$$

so that the mappings

$$x \to s.(x.\omega), \qquad x \to (s \circ x).\omega$$

are homomorphisms $Z(S)^+ \to \Omega$. To show that they coincide it suffices to consider the elements x in the generating set S of $Z(S)^+$. But for s, t of S we get†

$$s.(t.\omega) = s.(t\omega) = s(t\omega) = (s \circ t)\omega = (s \circ t).\omega.$$

We have shown that (2.3) holds for all y of $Z(S)^+$ and all x of S. Applying the same reasoning again we establish (2.3) by verifying that the mappings

$$x \to x.(y.\omega), \qquad x \to (x \circ y).\omega$$

are homomorphisms $Z(S)^+ \to \Omega$. This follows by (2.2) and A (iii).

† Here we use (2.1) and the S-group structure of Ω.

For (iv) we consider R^+ as an S-group. By (iii) we obtain a $Z(S)$-group structure such that $s.v = \theta(s)v$ for all s of S and all v of R^+. By (ii) there exists a unique epimorphism $\bar{\theta}$: $Z(S)^+ \to R^+$ with

$$\bar{\theta}(s) = \theta(s) \quad (s \in S).$$

For all v of R the mappings $x \to x.v$, $x \to \bar{\theta}(x)v$ are homomorphisms $Z(S)^+ \to R^+$ coinciding on S. Hence

$$x.v = \bar{\theta}(x)v \quad \text{for all } x \text{ of } Z(S), \text{ all } v \text{ of } R. \tag{2.4}$$

Considering the mappings

$$y \to \bar{\theta}(s \circ y), \qquad y \to s.\bar{\theta}(y)$$

we prove by the same type of reasoning that

$$\bar{\theta}(s \circ y) = s.\bar{\theta}(y) \quad \text{for all } s \text{ of } S, \text{ all } y \text{ of } Z(S). \tag{2.5}$$

Repeating the same procedure once more with the mappings

$$x \to \bar{\theta}(x \circ y), \qquad x \to x.\bar{\theta}(y),$$

we obtain from (2.5) $\qquad \bar{\theta}(x \circ y) = x.\bar{\theta}(y),$

which in conjunction with (2.4) gives

$$\bar{\theta}(x \circ y) = \bar{\theta}(x)\bar{\theta}(y)$$

and so establishes $\bar{\theta}$ as a near-ring epimorphism.

To prove the uniqueness assertion we consider a near-ring $\bar{R}$ generated by the distributive semigroup $\bar{S}$, where λ: $S \to \bar{S}$ is an isomorphism of semigroups. By (iv), λ extends to a near-ring epimorphism $\bar{\lambda}$: $Z(S) \to \bar{R}$.

If $\bar{R}^+$ is the free group on $\bar{S}$, then $\bar{\lambda}$ maps the free generators of $Z(S)^+$ bi-uniquely onto the free generators of $\bar{R}^+$, whence it is an isomorphism.

Next assume $\bar{R}$ to satisfy condition (iii), of course with appropriate changes in wording. Then Σ is an $(\bar{R}, \bar{S})$-group, and, for all x of $Z(S)$ and all σ of Σ, $(\bar{\lambda}x)\sigma = x\sigma$. Thus $\bar{\lambda}x = 0$ implies $x = 0$, and so $\bar{\lambda}$ is an isomorphism.

If $\bar{R}$ satisfies a condition analogous to (iv), we obtain a near-ring epimorphism ψ: $\bar{R} \to Z(S)$. But $\psi\bar{\lambda}$ is the identity mapping on the set S of generators of the additive group. Hence $\bar{\lambda}$ is an isomorphism. This completes the proof of Theorem 2.1.

The preceding theorem establishes, for every near-ring R generated by the distributive semigroup S, a canonical near-ring epimorphism $\bar{\theta}$: $Z(S) \to R$ whose kernel $\mathfrak{A}$ is an ideal in $Z(S)$. Thus R appears as residue near-ring $Z(S)/\mathfrak{A}$. If Ω is an (R, S)-group, then for all x of $Z(S)$ and all ω of Ω, $x\omega = \bar{\theta}(x)\omega$. If Ω is a $(Z(S), S)$-group, denote

by $\mathfrak{A}\Omega$ the minimal normal subgroup of Ω containing the elements $a\omega$ for all a of $\mathfrak{A}$ and all ω of Ω. We then have immediately

[2.2] *If Ω has an (R, S)-group structure μ and if μ' is the induced $(Z(S), S)$-group structure, then under μ', $\mathfrak{A}\Omega = 0$. Conversely, if Ω has a $(Z(S), S)$-group structure ν such that $\mathfrak{A}\Omega = 0$, then the mapping ν^*: $(v, w) \to \bar{v}w$, where, if $v \in R$, $\bar{v}$ is some element in $Z(S)$ with $\bar{\theta}(\bar{v}) = v$, defines an (R, S)-group structure on Ω which is independent of the choice of representatives $\bar{v}$. Also $(\nu^*)' = \nu$, $(\mu')^* = \mu$.*

3. Free sum and cartesian sum†

Throughout R is a near-ring generated by the distributive semigroup S. Unless otherwise mentioned, in the present section all groups are (R, S)-groups and their homomorphisms are R-homomorphisms.

Let Λ be an index set, and let $\{\Omega_\lambda\}$ $(\lambda \in \Lambda)$ be a family of (R, S)-groups. A *free (R, S)-sum* of the groups Ω_λ is given by an (R, S)-group Ω together with R-homomorphisms α^λ: $\Omega_\lambda \to \Omega$ $(\lambda \in \Lambda)$. The following postulates are to be satisfied:

D (i) *whenever Δ is an (R, S)-group and, for each λ of Λ, ϕ^λ: $\Omega_\lambda \to \Delta$ is an R-homomorphism, then there exists an R-homomorphism ϕ: $\Omega \to \Delta$ such that, for all λ, $\phi \circ \alpha^\lambda = \phi^\lambda$;*

(ii) *if ϕ, ψ are R-homomorphisms $\Omega \to \Delta$, and if, for all λ,*

$$\phi \circ \alpha^\lambda = \psi \circ \alpha^\lambda,$$

then $\phi = \psi$.

If $\nu, \lambda \in \Lambda$, let ι_ν^ν be the identity mapping on Ω_ν and ι_ν^λ $(\nu \neq \lambda)$ be the null mapping $\Omega_\lambda \to \Omega_\nu$. By D there exists a unique homomorphism η_ν: $\Omega \to \Omega_\nu$ with

$$\eta_\nu \circ \alpha^\lambda = \iota_\nu^\lambda \quad (\lambda \in \Lambda). \tag{3.1}$$

It follows that the α^λ are monomorphisms and the η_λ are epimorphisms. We shall then speak of the free (R, S)-sum Ω with *defining injections* α^λ, and *associated projections* η_λ.

If Δ is the free (R, S)-sum of groups Δ_μ $(\mu \in \mathrm{M})$ with defining injections β^μ, if f is a mapping $\Lambda \to \mathrm{M}$ and, for each λ, ϕ^λ: $\Omega_\lambda \to \Delta_{f(\lambda)}$ is an R-homomorphism, then we obtain an R-homomorphism ϕ: $\Omega \to \Delta$ with

$$\phi \circ \alpha^\lambda = \beta^{f(\lambda)} \circ \phi^\lambda \quad \text{for all } \lambda.$$

A *cartesian (R, S)-sum* of the groups Ω_λ is given by an (R, S)-group

† For a discussion of the free sum ('free product') and the cartesian sum ('direct product') see e.g. (**3**).

Ω together with R-homomorphisms $\pi_\lambda: \Omega \to \Omega_\lambda$ $(\lambda \in \Lambda)$. The following postulates are to be satisfied:

E (i) *whenever* Δ *is an* (R, S)*-group and, for each* λ *of* Λ, $\phi_\lambda: \Delta \to \Omega_\lambda$ *is an* R*-homomorphism, there exists an* R*-homomorphism* $\phi: \Delta \to \Omega$ *such that, for all* λ, $\pi_\lambda \circ \phi = \phi_\lambda$;

(ii) *if* ϕ, ψ *are* R*-homomorphisms* $\Delta \to \Omega$, *and if, for all* λ,

$$\pi_\lambda \circ \phi = \pi_\lambda \circ \psi,$$

then $\phi = \psi$.

There exist then homomorphisms $\beta^\nu: \Omega_\nu \to \Omega$ with

$$\pi_\lambda \circ \beta^\nu = \iota_\lambda^\nu \quad (\lambda \in \Lambda). \tag{3.2}$$

Thus π_λ is an epimorphism, β^λ is a monomorphism. We shall call Ω the *cartesian* (R, S)-sum of the groups Ω_λ with *defining projections* π_λ and *associated injections* β^λ.

If Δ is the cartesian (R, S)-sum of groups Δ_μ $(\mu \in \mathrm{M})$ with defining projections σ_μ, if f is a mapping $\mathrm{M} \to \Lambda$ and, for each μ of M, ϕ_μ: $\Omega_{f(\mu)} \to \Delta_\mu$ is an R-homomorphism, then we obtain an R-homomorphism $\phi: \Omega \to \Delta$ with $\sigma_\mu \circ \phi = \phi_\mu \circ \pi_{f(\mu)}$.

THEOREM 3.1. (i) *If* Ω, Ω' *are free* (R, S)*-sums of the groups* Ω_λ *with defining injections* α^λ, α'^λ, *then the* R*-homomorphism* $\theta: \Omega \to \Omega'$ *with* $\theta \circ \alpha^\lambda = \alpha'^\lambda$ *is an isomorphism*;

(ii) *if* Ω, Ω' *are cartesian* (R, S)*-sums of the groups* Ω_λ *with defining projections* π_λ, π'_λ, *then the* R*-homomorphism* $\theta: \Omega' \to \Omega$ *with* $\pi_\lambda \circ \theta = \pi'_\lambda$ *is an isomorphism.*

Proof. (i) There exists an R-homomorphism $\theta': \Omega' \to \Omega$ with

$$\theta' \circ \alpha'^\lambda = \alpha^\lambda.$$

We then get, for all λ, $\theta' \circ \theta \circ \alpha^\lambda = \iota \circ \alpha^\lambda$, ι being the identity mapping of Ω. By D (ii), $\theta' \circ \theta = \iota$. Similarly $\theta \circ \theta'$ is the identity mapping of Ω'. Hence θ is an isomorphism. (ii) follows by similar reasoning.

We can now speak of the *free* (R, S)-sum of the groups Ω_λ, denoting it by $\circledast(R, S)\Omega_\lambda$, specifying, if necessary, the index set Λ and the defining injections; if no confusion is possible, we shall also use the symbol $\circledast\Omega_\lambda$. Similarly we denote the cartesian sum by $+(R, S)\Omega_\lambda$ or $+\Omega_\lambda$.

Let M, Λ be index sets, $\{\Omega_\lambda\}$ $(\lambda \in \Lambda)$ be a family of (R, S)-groups, and let $\{\Lambda_\mu\}$ be a partition of Λ into disjoint sets with index set M.

[3.2] (i) *Let* $\Omega = \circledast\Omega_\lambda$ $(\lambda \in \Lambda)$ *be defined by the injections* α^λ. *For* μ *in* M *let* $\Omega_{(\mu)}$ *be the subgroup of* Ω *generated by* $\mathrm{Im}\,\alpha^\lambda$ *(all* $\lambda \in \Lambda_\mu$*). Then* $\Omega = \circledast\Omega_{(\mu)}$ $(\mu \in \mathrm{M})$ *is defined by the natural injections* $\beta^{(\mu)}: \Omega_{(\mu)} \to \Omega$, *and*

for each μ in M, $\Omega_{(\mu)} = \circledast\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$ *is defined by the unique injections* $\alpha^\lambda_{(\mu)}$ *with* $\beta^{(\mu)} \circ \alpha^\lambda_{(\mu)} = \alpha^\lambda$ $(\lambda \in \Lambda_\mu)$.

(ii) *Let* $\Omega = \dotplus\Omega_\lambda$ $(\lambda \in \Lambda)$ *be defined by the projections* π_λ. *For* μ *in* M *let* $\Omega_{(\mu)}$ *be the quotient group of* Ω *modulo the intersection of* $\operatorname{Ker}\pi_\lambda$ *(all* $\lambda \in \Lambda_\mu$). *Then* $\Omega = \dotplus\Omega_{(\mu)}$ $(\mu \in \mathrm{M})$ *is defined by the natural projections* $\kappa_{(\mu)}: \Omega \to \Omega_{(\mu)}$, *and, for each* μ *in* M, $\Omega_{(\mu)} = \dotplus\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$ *is defined by the unique projections* $\pi^{(\mu)}_\lambda$ *with* $\pi^{(\mu)}_\lambda \circ \kappa_{(\mu)} = \pi_\lambda$ $(\lambda \in \Lambda_\mu)$.

Proof. We prove only (i). Together with $\operatorname{Im}\alpha^\lambda$, for all λ, $\Omega_{(\mu)}$ is also an (R, S)-group for all μ, and $\beta^{(\mu)}$, $\alpha^\lambda_{(\mu)}$ are R-homomorphisms.

Let $\mu \in \mathrm{M}$. Assume that, for each λ in Λ_μ, $\phi^\lambda: \Omega_\lambda \to \Delta$ is an R-homomorphism. For λ not in Λ_μ take $\phi^\lambda: \Omega_\lambda \to \Delta$ to be the null homomorphism. By D (i) we obtain an R-homomorphism $\phi: \Omega \to \Delta$ with $\phi \circ \alpha^\lambda = \phi^\lambda$ for all λ. Thus for all λ in Λ_μ $(\phi \circ \beta^{(\mu)}) \circ \alpha^\lambda_{(\mu)} = \phi^\lambda$. The uniqueness postulate D (ii) follows from the fact that the groups $\operatorname{Im}\alpha^\lambda_{(\mu)}$ $(\lambda \in \Lambda_\mu)$ generate $\Omega_{(\mu)}$. Thus $\Omega_{(\mu)} = \circledast\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$.

Next, for each μ in M, let $\phi^{(\mu)}: \Omega_{(\mu)} \to \Delta$ be an R-homomorphism. There exists an R-homomorphism $\phi: \Omega \to \Delta$ with $\phi \circ \alpha^\lambda = \phi^{(\mu)} \circ \alpha^\lambda_{(\mu)}$ for all λ in Λ, $\mu = \mu(\lambda)$ being chosen so that $\lambda \in \Lambda_\mu$. Hence for each μ in M and all λ in Λ_μ,

$$(\phi \circ \beta^{(\mu)}) \circ \alpha^\lambda_{(\mu)} = \phi^{(\mu)} \circ \alpha^\lambda_{(\mu)};$$

since $\Omega_{(\mu)} = \circledast\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$, we have $\phi \circ \beta^{(\mu)} = \phi^{(\mu)}$. Finally, if ϕ, ψ are R-homomorphisms, $\Omega \to \Delta$ with $\phi \circ \beta^{(\mu)} = \psi \circ \beta^{(\mu)}$ for all μ in M, then $\phi \circ \alpha^\lambda = \psi \circ \alpha^\lambda$ for all λ in Λ and so $\phi = \psi$.

[3.3] (i) D (ii) *can be replaced by*

D (ii′) *The groups* $\operatorname{Im}\alpha^\lambda$ *generate* Ω.

(ii) E (ii) *can be replaced by*

E (ii′) *The intersection of the groups* $\operatorname{Ker}\pi_\lambda$ *is* 0.

Proof of (i). D (ii′) clearly implies D (ii). For the converse let Ω satisfy D (i), (ii). Apply [3.2] with

$$\mathrm{M} = \{1\}, \qquad \Lambda = \Lambda_1.$$

$\Omega_{(1)}$ is the group generated by $\operatorname{Im}\alpha^\lambda$ $(\lambda \in \Lambda)$. By Theorem 3.1, however, the injection $\Omega_{(1)} \to \Omega$ is an isomorphism.

THEOREM 3.4. *Every family* $\{\Omega_\lambda\}$ *of* (R, S)*-groups has a free* (R, S)*-sum and a cartesian* (R, S)*-sum.*

Proof. (i) We take first $R = Z(S)$. Let Ω be the 'abstract' free sum, i.e. the free $(Z, 1)$-sum of the groups Ω_λ. For s in S, and for each λ in Λ, $\omega \to s\omega$ is an endomorphism of Ω_λ. If α^λ are the defining

injections of the free $(Z, 1)$-sum, we get by D (i) an endomorphism of Ω, again denoted by s such that

$$s(\alpha^\lambda \omega) = \alpha^\lambda(s\omega) \quad (\text{all } \lambda \text{ in } \Lambda, \text{ all } \omega \text{ in } \Omega_\lambda). \tag{3.3}$$

If $s, t \in S$, we verify that for all λ in Λ, all ω in Ω_λ,

$$(st)(\alpha^\lambda\omega) = s(t(\alpha^\lambda\omega)),$$

whence by D (ii) (for the the free $(Z, 1)$-sum Ω), Ω is an S-group, and so, by Theorem 2.1, an (R, S)-group, and the α^λ are R-homomorphisms.

Now, for each λ, let ϕ^λ: $\Omega_\lambda \to \Delta$ be an R-homomorphism. There exists a homomorphism ϕ: $\Omega \to \Delta$ with $\phi \circ \alpha^\lambda = \phi^\lambda$, for all λ. If $s \in S$, then for all λ, all ω in Ω_λ,

$$\phi(s\alpha^\lambda\omega) = \phi(\alpha^\lambda s\omega) = \phi^\lambda(s\omega) = s(\phi^\lambda\omega) = s(\phi \circ \alpha^\lambda\omega).$$

By D (ii) (for the free $(Z, 1)$-sum Ω) $\phi s = s\phi$. Hence ϕ is an S-homomorphism, i.e. an R-homomorphism. Thus Ω satisfies postulate D (i), and trivially, of course, D (ii), for a free (R, S)-sum.

For an arbitrary d.g. near-ring R we use the canonical epimorphism θ: $Z(S) \to R$ with kernel $\mathfrak{A}$, as in § 2. By Theorem 2.1 the groups Ω_λ are $(Z(S), S)$-groups. We have established the existence of their free $(Z(S), S)$-sum, which we now denote by $\bar{\Omega}$, with defining injections $\bar{\alpha}^\lambda$. Let $\Omega = \bar{\Omega}/\mathfrak{A}\bar{\Omega}$ and let ψ: $\bar{\Omega} \to \Omega$ be the natural projection. Ω is an (R, S)-group, ψ is an S-homomorphism and so, for all λ, $\alpha^\lambda = \psi \circ \bar{\alpha}^\lambda$ is an S-homomorphism and so an R-homomorphism. Postulate D (ii′) for Ω follows by D (ii′) for $\bar{\Omega}$.

For each λ let ϕ^λ: $\Omega_\lambda \to \Delta$ be an R-homomorphism. If Ω_λ, Δ are considered as $Z(S)$-groups, ϕ^λ is a $Z(S)$-homomorphism. Hence there exists a $Z(S)$-homomorphism $\bar{\phi}$: $\bar{\Omega} \to \Delta$ with $\bar{\phi} \circ \bar{\alpha}^\lambda = \phi^\lambda$. Since $\mathfrak{A}\Delta = 0$ by [2.2], $\operatorname{Ker}\bar{\phi} \supseteq \mathfrak{A}\bar{\Omega}$. Hence there exists a $Z(S)$-homomorphism ϕ: $\Omega \to \Delta$ with $\bar{\phi} = \phi \circ \psi$. This implies $\phi \circ \alpha^\lambda = \phi^\lambda$ for all λ. But, since Ω and Δ are R-groups, ϕ is in fact an R-homomorphism.

(ii) We first construct the cartesian $(Z, 1)$-sum Ω of the groups Ω_λ in the usual manner. The elements ω of Ω are those mappings of Λ into the set union of the Ω_λ with $\omega(\lambda) \in \Omega_\lambda$ for all λ. Addition is defined by

$$(\omega_1 + \omega_2)(\lambda) = \omega_1(\lambda) + \omega_2(\lambda) \quad \text{for all } \lambda.$$

We now endow Ω with an R-group structure by writing $[x\omega](\lambda) = x(\omega(\lambda))$ for all λ. It is then easily verified that this construction leads to the cartesian (R, S)-sum.

We have seen that $\textstyle\sum(Z, 1)\Omega_\lambda$ is the underlying group of $\textstyle\sum(R, S)\Omega_\lambda$ while the analogous result is not in general true for free sums.

for each μ in M, $\Omega_{(\mu)} = \ast\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$ *is defined by the unique injections* $\alpha^\lambda_{(\mu)}$ *with* $\beta^{(\mu)} \circ \alpha^\lambda_{(\mu)} = \alpha^\lambda$ $(\lambda \in \Lambda_\mu)$.

(ii) *Let* $\Omega = +\Omega_\lambda$ $(\lambda \in \Lambda)$ *be defined by the projections* π_λ. *For* μ *in* M *let* $\Omega_{(\mu)}$ *be the quotient group of* Ω *modulo the intersection of* $\operatorname{Ker}\pi_\lambda$ *(all* $\lambda \in \Lambda_\mu$*). Then* $\Omega = +\Omega_{(\mu)}$ $(\mu \in \mathrm{M})$ *is defined by the natural projections* $\kappa_{(\mu)}\colon \Omega \to \Omega_{(\mu)}$, *and, for each* μ *in* M, $\Omega_{(\mu)} = +\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$ *is defined by the unique projections* $\pi^{(\mu)}_\lambda$ *with* $\pi^{(\mu)}_\lambda \circ \kappa_{(\mu)} = \pi_\lambda$ $(\lambda \in \Lambda_\mu)$.

Proof. We prove only (i). Together with $\operatorname{Im}\alpha^\lambda$, for all λ, $\Omega_{(\mu)}$ is also an (R, S)-group for all μ, and $\beta^{(\mu)}$, $\alpha^\lambda_{(\mu)}$ are R-homomorphisms.

Let $\mu \in \mathrm{M}$. Assume that, for each λ in Λ_μ, $\phi^\lambda\colon \Omega_\lambda \to \Delta$ is an R-homomorphism. For λ not in Λ_μ take $\phi^\lambda\colon \Omega_\lambda \to \Delta$ to be the null homomorphism. By D (i) we obtain an R-homomorphism $\phi\colon \Omega \to \Delta$ with $\phi \circ \alpha^\lambda = \phi^\lambda$ for all λ. Thus for all λ in Λ_μ $(\phi \circ \beta^{(\mu)}) \circ \alpha^\lambda_{(\mu)} = \phi^\lambda$. The uniqueness postulate D (ii) follows from the fact that the groups $\operatorname{Im}\alpha^\lambda_{(\mu)}$ $(\lambda \in \Lambda_\mu)$ generate $\Omega_{(\mu)}$. Thus $\Omega_{(\mu)} = \ast\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$.

Next, for each μ in M, let $\phi^{(\mu)}\colon \Omega_{(\mu)} \to \Delta$ be an R-homomorphism. There exists an R-homomorphism $\phi\colon \Omega \to \Delta$ with $\phi \circ \alpha^\lambda = \phi^{(\mu)} \circ \alpha^\lambda_{(\mu)}$ for all λ in Λ, $\mu = \mu(\lambda)$ being chosen so that $\lambda \in \Lambda_\mu$. Hence for each μ in M and all λ in Λ_μ,

$$(\phi \circ \beta^{(\mu)}) \circ \alpha^\lambda_{(\mu)} = \phi^{(\mu)} \circ \alpha^\lambda_{(\mu)};$$

since $\Omega_{(\mu)} = \ast\Omega_\lambda$ $(\lambda \in \Lambda_\mu)$, we have $\phi \circ \beta^{(\mu)} = \phi^{(\mu)}$. Finally, if ϕ, ψ are R-homomorphisms, $\Omega \to \Delta$ with $\phi \circ \beta^{(\mu)} = \psi \circ \beta^{(\mu)}$ for all μ in M, then $\phi \circ \alpha^\lambda = \psi \circ \alpha^\lambda$ for all λ in Λ and so $\phi = \psi$.

[3.3] (i) D (ii) *can be replaced by*

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Proof of (i). D (ii′) clearly implies D (ii). For the converse let Ω satisfy D (i), (ii). Apply [3.2] with

$$\mathrm{M} = \{1\}, \qquad \Lambda = \Lambda_1.$$

$\Omega_{(1)}$ is the group generated by $\operatorname{Im}\alpha^\lambda$ $(\lambda \in \Lambda)$. By Theorem 3.1, however, the injection $\Omega_{(1)} \to \Omega$ is an isomorphism.

THEOREM 3.4. *Every family* $\{\Omega_\lambda\}$ *of* (R, S)*-groups has a free* (R, S)*-sum and a cartesian* (R, S)*-sum.*

Proof. (i) We take first $R = Z(S)$. Let Ω be the 'abstract' free sum, i.e. the free $(Z, 1)$-sum of the groups Ω_λ. For s in S, and for each λ in Λ, $\omega \to s\omega$ is an endomorphism of Ω_λ. If α^λ are the defining

injections of the free $(Z, 1)$-sum, we get by D (i) an endomorphism of Ω, again denoted by s such that

$$s(\alpha^\lambda \omega) = \alpha^\lambda(s\omega) \quad (\text{all } \lambda \text{ in } \Lambda, \text{ all } \omega \text{ in } \Omega_\lambda). \tag{3.3}$$

If $s, t \in S$, we verify that for all λ in Λ, all ω in Ω_λ,

$$(st)(\alpha^\lambda \omega) = s(t(\alpha^\lambda \omega)),$$

whence by D (ii) (for the the free $(Z, 1)$-sum Ω), Ω is an S-group, and so, by Theorem 2.1, an (R, S)-group, and the α^λ are R-homomorphisms.

Now, for each λ, let $\phi^\lambda\colon \Omega_\lambda \to \Delta$ be an R-homomorphism. There exists a homomorphism $\phi\colon \Omega \to \Delta$ with $\phi \circ \alpha^\lambda = \phi^\lambda$, for all λ. If $s \in S$, then for all λ, all ω in Ω_λ,

$$\phi(s\alpha^\lambda \omega) = \phi(\alpha^\lambda s\omega) = \phi^\lambda(s\omega) = s(\phi^\lambda \omega) = s(\phi \circ \alpha^\lambda \omega).$$

By D (ii) (for the free $(Z, 1)$-sum Ω) $\phi s = s\phi$. Hence ϕ is an S-homomorphism, i.e. an R-homomorphism. Thus Ω satisfies postulate D (i), and trivially, of course, D (ii), for a free (R, S)-sum.

For an arbitrary d.g. near-ring R we use the canonical epimorphism $\theta\colon Z(S) \to R$ with kernel $\mathfrak{A}$, as in § 2. By Theorem 2.1 the groups Ω_λ are $(Z(S), S)$-groups. We have established the existence of their free $(Z(S), S)$-sum, which we now denote by $\bar{\Omega}$, with defining injections $\bar{\alpha}^\lambda$. Let $\Omega = \bar{\Omega}/\mathfrak{A}\bar{\Omega}$ and let $\psi\colon \bar{\Omega} \to \Omega$ be the natural projection. Ω is an (R, S)-group, ψ is an S-homomorphism and so, for all λ, $\alpha^\lambda = \psi \circ \bar{\alpha}^\lambda$ is an S-homomorphism and so an R-homomorphism. Postulate D(ii′) for Ω follows by D (ii′) for $\bar{\Omega}$.

For each λ let $\phi^\lambda\colon \Omega_\lambda \to \Delta$ be an R-homomorphism. If Ω_λ, Δ are considered as $Z(S)$-groups, ϕ^λ is a $Z(S)$-homomorphism. Hence there exists a $Z(S)$-homomorphism $\bar{\phi}\colon \bar{\Omega} \to \Delta$ with $\bar{\phi} \circ \bar{\alpha}^\lambda = \phi^\lambda$. Since $\mathfrak{A}\Delta = 0$ by [2.2], $\operatorname{Ker}\bar{\phi} \supseteq \mathfrak{A}\bar{\Omega}$. Hence there exists a $Z(S)$-homomorphism ϕ: $\Omega \to \Delta$ with $\bar{\phi} = \phi \circ \psi$. This implies $\phi \circ \alpha^\lambda = \phi^\lambda$ for all λ. But, since Ω and Δ are R-groups, ϕ is in fact an R-homomorphism.

(ii) We first construct the cartesian $(Z, 1)$-sum Ω of the groups Ω_λ in the usual manner. The elements ω of Ω are those mappings of Λ into the set union of the Ω_λ with $\omega(\lambda) \in \Omega_\lambda$ for all λ. Addition is defined by

$$(\omega_1 + \omega_2)(\lambda) = \omega_1(\lambda) + \omega_2(\lambda) \quad \text{for all } \lambda.$$

We now endow Ω with an R-group structure by writing $[x\omega](\lambda) = x(\omega(\lambda))$ for all λ. It is then easily verified that this construction leads to the cartesian (R, S)-sum.

We have seen that $\boldsymbol{+}(Z, 1)\Omega_\lambda$ is the underlying group of $\boldsymbol{+}(R, S)\Omega_\lambda$ while the analogous result is not in general true for free sums.

Examples. (1) If R is a ring with identity e, then every (R, R^*) group on which e acts as the identity mapping, is abelian. For

$$(-e)(\omega_1+\omega_2) = (-e)\omega_1+(-e)\omega_2$$

implies $\omega_1+\omega_2 = \omega_2+\omega_1$. The free (R, R^*) sum of such groups is thus their direct sum.

(2) Let $Z/(n)$ be the residue ring of integers to modulus n. Every group Ω whose exponent divides n is a $(Z/(n), 1)$ group. The free $(Z/(n), 1)$ sum of such groups is their free sum reduced by the relation $n\omega = 0$.

(3) Let S be the multiplicative semigroup consisting of the powers of n. The free (Z, S) sum of (Z, S) groups is their free sum reduced by the relations

$$n(\omega_1+\omega_2) = n\omega_1+n\omega_2.$$

(4) Let S be a multiplicative group and let Ω be the free $(Z(S), S)$ sum of copies of $Z(S)^+$. Ω contains a set Δ of free S generators, i.e. a set Δ of elements such that Ω is the free group on the elements $s\delta$ (all $s \in S$, all $\delta \in \Delta$). Every mapping of Δ into an S group Σ on which S acts as a group of automorphisms can be extended to an S homomorphism $\Omega \to \Sigma$, and every such group Σ can be represented as a quotient group of a 'free S group' Ω.

4. Orthogonal sums

All groups considered are again (R, S) groups and their homomorphisms are R homomorphisms. $\{\Omega_\lambda\}$ $(\lambda \in \Lambda)$ is a fixed family of such groups. An *orthogonal sum* Ω of the groups Ω_λ with *defining injections* β^λ and *defining projections* τ_λ is given by an (R, S) group Ω, together with R homomorphisms $\beta^\lambda: \Omega_\lambda \to \Omega$, $\tau_\lambda: \Omega \to \Omega_\lambda$ such that

$$\tau_\nu \circ \beta^\lambda = \iota_\nu^\lambda. \tag{4.1}$$

Thus the τ_λ are in fact projections, the β^λ injections. If Ω' is another orthogonal sum of the groups Ω_λ, defined by projections τ'_λ and injections β'^λ, and if the R homomorphism $\theta: \Omega \to \Omega'$ satisfies

$$\tau'_\lambda \circ \theta = \tau_\lambda, \quad \theta \circ \beta^\lambda = \beta'^\lambda \quad \text{for all } \lambda, \tag{4.2}$$

θ will be called an *orthogonal homomorphism*. The products and inverses of such homomorphisms are again orthogonal, if defined.

For the given orthogonal sum Ω let $C(\Omega)$ be the subgroup generated by the groups $\operatorname{Im}\beta^\lambda$ and denote by $C(\beta^\lambda): \Omega_\lambda \to C(\Omega)$, $C(\tau_\lambda): C(\Omega) \to \Omega_\lambda$ the homomorphisms induced by β^λ, τ_λ respectively. Let further

$$F(\Omega) = \Omega \Big/ \bigcap_\lambda \operatorname{Ker}\tau_\lambda$$

and denote by

$$F(\beta^\lambda)\colon \Omega_\lambda \to F(\Omega), \qquad F(\pi_\lambda)\colon F(\Omega) \to \Omega_\lambda$$

the induced homomorphisms. With these homomorphisms $C(\Omega)$ and $F(\Omega)$ are again orthogonal sums of the groups Ω_λ and the injection $i_\Omega\colon C(\Omega) \to \Omega$ and the projection $p_\Omega\colon \Omega \to F(\Omega)$ are orthogonal. If i_Ω is an isomorphism, Ω is said to be *covered*; if p_Ω is an isomorphism, Ω is said to be *filtered*. Thus $C(\Omega)$ is covered, $F(\Omega)$ is filtered. If $\theta\colon \Omega \to \Omega'$ is an orthogonal homomorphism, then there exist unique orthogonal homomorphisms

$$C(\theta)\colon C(\Omega) \to C(\Omega'), \qquad F(\theta)\colon F(\Omega) \to F(\Omega')$$

such that the diagram

$$\begin{tikzcd} C(\Omega) \arrow[r, "C(\theta)"] \arrow[d, "i_\Omega"'] & C(\Omega') \arrow[d, "i_{\Omega'}"] \\ \Omega \arrow[r, "\theta"] \arrow[d, "p_\Omega"'] & \Omega' \arrow[d, "p_{\Omega'}"] \\ F(\Omega) \arrow[r, "F(\theta)"] & F(\Omega)' \end{tikzcd} \tag{4.3}$$

is commutative. From the definitions it follows that $C(\theta)$ is always an epimorphism and is an isomorphism if θ is a monomorphism; $F(\theta)$ is always a monomorphism and is an isomorphism if θ is an epimorphism.

In [4.1] all orthogonal sums are sums of the groups Ω_λ.

[4.1] (i) *The free (R, S)-sum Ω of the groups Ω_λ with its defining injections α^λ and associated projections η_λ is a covered orthogonal sum, and*

(*a*) *for every covered orthogonal sum Ω' there exists a unique orthogonal epimorphism $\Omega \to \Omega'$;*

(*b*) *for every orthogonal sum Ω' there exists a unique orthogonal homomorphism $\Omega \to \Omega'$.*

A covered orthogonal sum Ω is a free (R, S)-sum if it satisfies (*a*) *or* (*b*).

(ii) *The cartesian (R, S)-sum Ω of the groups Ω_λ with its associated injections β^λ and its defining projections π_λ is a filtered orthogonal sum, and*

(*a*) *for every filtered orthogonal sum Ω' there exists a unique orthogonal monomorphism $\Omega' \to \Omega$;*

(*b*) *for every orthogonal sum Ω' there exists a unique orthogonal homomorphism $\Omega' \to \Omega$.*

A filtered orthogonal sum Ω is a cartesian (R, S)-sum if it satisfies (*a*) *or* (*b*).

Proof of (i). By (3.1) and [3.3], $\Omega = \ast\Omega_\lambda$ is a covered orthogonal sum. If Ω' is an orthogonal sum with injections α'^λ and projections η'_λ, there exists a unique R-homomorphism $\theta\colon \Omega \to \Omega'$ with $\theta \circ \alpha^\lambda = \alpha'^\lambda$ for all λ. Then

$$\eta_\nu \circ \alpha^\lambda = \iota_\nu^\lambda = \eta'_\nu \circ \alpha'^\lambda = \eta'_\nu \circ \theta \circ \alpha^\lambda,$$

whence, by D (ii), $\eta_\lambda = \eta'_\lambda \circ \theta$ for all λ. Thus θ is orthogonal and (b) holds. If Ω' is covered, then $i_{\Omega'}$, i_Ω are isomorphisms; we have seen that $C(\theta)$ is an epimorphism, and so by the commutativity of (4.3), θ is an epimorphism.

If Ω is covered and satisfies (b), so that we get an orthogonal homomorphism $\theta\colon \Omega \to \Omega'$, then $\operatorname{Im}\theta = C(\Omega')$; this implies that Ω satisfies (a). Apply (a) to get orthogonal epimorphisms

$$\Omega \to \ast\Omega_\lambda, \qquad \ast\Omega_\lambda \to \Omega$$

whose composite mapping is the identity mapping $\Omega \to \Omega$. Hence the epimorphism $\Omega \to \ast\Omega_\lambda$ is an isomorphism.

By either (i) or (ii) in [4.1], or directly from the definitions, there exists a unique orthogonal homomorphism $\ast\Omega_\lambda \to +\Omega_\lambda$, which we shall denote throughout this section by ψ.

[4.2] *If Ω is an orthogonal sum of the groups Ω_λ and if*

$$\theta\colon \ast\Omega_\lambda \to \Omega, \qquad \phi\colon \Omega \to +\Omega_\lambda$$

are the orthogonal homomorphisms associated with Ω, then the diagram

$$\begin{tikzcd} \ast\Omega_\lambda \arrow[rr, "\psi"] \arrow[dr, "\theta"'] & & +\Omega_\lambda \\ & \Omega \arrow[ur, "\phi"'] & \end{tikzcd}$$

is commutative. Conversely, if this diagram is commutative for given R-homomorphisms θ, ϕ, then there exist unique homomorphisms $\Omega_\lambda \to \Omega$, $\Omega \to \Omega_\lambda$, so that Ω is an orthogonal sum and θ, ϕ are orthogonal.

Proof. If θ, ϕ are orthogonal, then so is $\phi \circ \theta$, whence by uniqueness $\psi = \phi \circ \theta$. If the diagram is commutative for given R-homomorphisms ϕ, θ, then we take $\theta \circ \alpha^\lambda$ as defining injections, $\pi_\lambda \circ \phi$ as defining projections of Ω. Uniqueness is immediate.

It is also possible to characterize the *direct (R, S)-sum* $\sum \Omega_\lambda$ in terms of orthogonal sums. Assuming that we have defined the direct (R, S)-sum by the usual construction of a direct sum endowed with the natural (R, S)-group structure, we get

[4.3]. *Let Ω be the direct (R, S)-sum of the groups Ω_λ. Then*

(a) Ω is a filtered, covered orthogonal sum;

(b) if Ω' is a filtered orthogonal sum, there exists a unique orthogonal monomorphism $\Omega \to \Omega'$;

(c) if Ω' is a covered orthogonal sum, there exists a unique orthogonal epimorphism $\Omega' \to \Omega$.

Either (a), or (b), or (c) characterizes Ω uniquely to within orthogonal isomorphism.

[4.4] *Let Ω be an orthogonal sum. Ω is covered if and only if there exists a sequence*

$$\ast\Omega_\lambda \longrightarrow \Omega \longrightarrow \sum \Omega_\lambda$$

of orthogonal homomorphisms, and Ω is filtered if and only if there exists a sequence

$$\sum \Omega_\lambda \longrightarrow \Omega \longrightarrow +\Omega_\lambda$$

of orthogonal homomorphisms.

[4.3] and [4.4] are stated only for completeness' sake; the proofs will be omitted. I only mention here that, by [4.1] and [4.3], $\sum \Omega_\lambda$ can be represented in either of the forms $F(\ast\Omega_\lambda)$ or $C(+\Omega_\lambda)$.

It is well known that, if the index set Λ is finite, then $\sum \Omega_\lambda = +\Omega_\lambda$. On the other hand we have seen in § 3, Example 1, that $\sum \Omega_\lambda = \ast\Omega_\lambda$ is also possible.

5. Free and projective groups

A subset Ω' of an (R, S)-group Ω is called a *free (R, S)-basis*, or simply a *free basis* of Ω if for every (R, S)-group Δ and for every mapping $f\colon \Omega' \to \Delta$ there exists a unique R-homomorphism $\phi\colon \Omega \to \Delta$ whose restriction to Ω' is f. An (R, S)-group which has a free basis is said to be *free*.

For any set Δ' of elements of an R-group Δ, we denote by $\{\Delta'\}_R$ the minimal R-subgroup Δ_1 of Δ containing Δ'; we say also that Δ' is an *R-generating set* of Δ_1.

To construct free (R, S)-groups we return to the group $\Sigma(S)$ considered in § 2. We verify that this group has $[u]$ as free $(Z(S), S)$-basis. Let $R = Z(S)/\mathfrak{A}$ be the canonical representation of R derived in § 2. Put

$$\Sigma(R, S) = \Sigma(S)/\mathfrak{A}\Sigma(S).$$

This is an (R, S)-group by (2.2). Denote by τ the image of $[u]$ in $\Sigma(R, S)$. Then

$$\{\tau\}_R = \Sigma(R, S).$$

If now Δ is an (R, S)-group, δ an element in Δ, then viewing Δ as a

$(Z(S), S)$-group we get an S-homomorphism ϕ^*: $\Sigma(S) \to \Delta$ with $\phi^*[u] = \delta$. But $\mathfrak{A}\Delta = 0$, whence $\mathfrak{A}\Sigma(S) \subseteq \operatorname{Ker}\phi^*$. Thus ϕ^* induces an R-homomorphism ϕ: $\Sigma(R, S) \to \Delta$ with $\phi\tau = \delta$. We have shown that $\Sigma(R, S)$ has the free basis τ.

THEOREM 5.1. *Let Ω' be a non-empty set of elements in the (R, S)-group Ω. Ω' is a free basis of Ω if and only if*

(i) *for each ω in Ω' there exists an R-isomorphism*

$$\{\omega\}_R \simeq \Sigma(R, S)$$

with ω being mapped onto τ;

(ii) *Ω is the free (R, S)-sum of the groups $\{\omega\}_R$ for ω in Ω'.*

In particular Ω is a free (R, S)-group if and only if $\Omega = 0$ or Ω is the free (R, S)-sum of R-isomorphic copies of $\Sigma(R, S)$.

Proof. We prove first the lemma:

LEMMA 5.2. *If Ω is the free (R, S)-sum of (R, S)-groups Ω_λ with defining injections α^λ and if, for each λ, Ω'_λ is a free basis of Ω_λ, then the set union Ω' of the sets $\alpha^\lambda\Omega'_\lambda$ is a free basis of Ω.*

This, in conjunction with the preceding result on $\Sigma(R, S)$, establishes one half of the theorem.

Assume that, in the situation of Lemma 5.2, Δ is an (R, S)-group and f a mapping $\Omega' \to \Delta$. Then there exist R-homomorphisms ϕ^λ: $\Omega_\lambda \to \Delta$, coinciding on Ω'_λ with $f \circ \alpha^\lambda$. If ϕ: $\Omega \to \Delta$ is the R-homomorphism with $\phi \circ \alpha^\lambda = \phi^\lambda$ for all λ, then ϕ coincides with f on $\alpha^\lambda\Omega'_\lambda$ for all λ, i.e. on Ω'. Two R-homomorphisms ϕ, ψ which coincide on Ω', i.e. on $\alpha^\lambda\Omega'_\lambda$ for all λ, lead to homomorphisms $\phi \circ \alpha^\lambda$, $\psi \circ \alpha^\lambda$ coinciding on Ω'_λ. By hypothesis $\phi \circ \alpha^\lambda = \psi \circ \alpha^\lambda$ for all λ, and so $\phi = \psi$. This completes the proof of Lemma 5.2.

For the converse we observe that for each ω in Ω' there exists an R-homomorphism ϕ_ω: $\Omega \to \Sigma(R, S)$ with $\phi_\omega \omega = \tau$, $\phi_\omega \omega' = 0$ if $\omega' \in \Omega'$, $\omega' \neq \omega$. The restriction ψ_ω of ϕ_ω to $\{\omega\}_R$ is an R-homomorphism. Considering ψ_ω in conjunction with the R-homomorphism

$$\theta_\omega: \Sigma(R, S) \to \{\omega\}_R$$

determined by $\theta_\omega \tau = \omega$ we find that ψ_ω is an isomorphism.

Now, for each ω in Ω', let ϕ_ω be an R-homomorphism $\{\omega\}_R \to \Delta$. Take f: $\Omega' \to \Delta$ to be the mapping given by $f\omega = \phi_\omega \omega$, and let ϕ: $\Omega \to \Delta$ be the homomorphism determined by f. If α^ω are the injections $\{\omega\}_R \to \Omega$, we get $(\phi \circ \alpha^\omega)\omega = \phi_\omega \omega$. But we have already seen that ω is a free basis of $\{\omega\}_R$, and so $\phi \circ \alpha^\omega = \phi_\omega$ for all $\omega \in \Omega'$. If now ϕ, ψ:

$\Omega \to \Delta$ are R-homomorphisms with $\phi \circ \alpha^{\omega} = \psi \circ \alpha^{\omega}$ for all ω in Ω', then in particular ϕ and ψ coincide on Ω', i.e. $\phi = \psi$.

In the usual manner one proves the theorem:

THEOREM 5.3. *Every (R, S)-group Ω has a representation*

$$\psi\colon \bar{\Omega} \to \Omega,$$

where $\bar{\Omega}$ is a free (R, S)-group and ψ is an R-epimorphism.

An (R, S)-group Ω is said to be *projective* if every diagram of (R, S)-groups and R-homomorphisms

\begin{tikzcd}
 & \Omega \arrow[d, "\phi"] \\
\Sigma \arrow[r, "\psi"'] & \Delta
\end{tikzcd} (5.1)

in which ψ is an epimorphism, can be completed to a commutative diagram

\begin{tikzcd}
 & \Omega \arrow[d, "\phi"] \arrow[dl, "\theta"'] \\
\Sigma \arrow[r, "\psi"'] & \Delta
\end{tikzcd} (5.2)

We have the theorem:

THEOREM 5.4. *The free (R, S)-sum Ω of (R, S)-groups Ω_λ is projective if and only if Ω_λ is projective for all λ.*

The proof follows at once from the definition of a free sum.

Following Baer we call an R-subgroup Δ of an (R, S)-group Ω an *R-retract* of Ω if $\Delta = \operatorname{Im} g$, where g is an idempotent R-endomorphism of Ω. As in module theory we get the theorem:

THEOREM 5.5. *A free (R, S)-group is projective and an R-retract of a projective (R, S)-group is projective. Every projective (R, S)-group is an R-retract of a free (R, S)-group.*

From now on we take the near-ring R to have a (multiplicative) identity e. We assume e to lie in the distributive semi-group S generating R. This imposes no restriction on the structure of R [cf. (**1**) 1.3.1]; for the R-groups Ω considered it amounts to the assumption that e induces an endomorphism on Ω.

In any (R, S)-group Ω the mapping $\eta_\Omega\colon \omega \to e\omega$ is an idempotent R-endomorphism, so that $\operatorname{Ker} \eta_\Omega$ is complemented by $\operatorname{Im} \eta_\Omega = e\Omega$. If $\phi\colon \Omega \to \Delta$ is an R-homomorphism, then $\phi \circ \eta_\Omega = \eta_\Delta \circ \phi$. An (R, S)-group Ω is *unitary*† if $\Omega = e\Omega$; $e\Omega$ is always the maximal unitary

† In [(**2**) 99] I called such groups 'regular'; the term 'unitary' seems more appropriate however.

R-subgroup of Ω, and Ker η_Ω is the maximal R-subgroup of Ω annihilated by R.

[5.6] *The free (R, S)-sum Ω of unitary (R, S)-groups Ω_λ is unitary.*

Proof. Write $\eta_{\Omega_\lambda} = \eta_\lambda$, denote the injections $\Omega_\lambda \to \Omega$ by α^λ. Then

$$\eta_\Omega \circ \alpha^\lambda = \alpha^\lambda \circ \eta_\lambda$$

for all λ. But $\eta_\lambda = \iota_\lambda^\lambda$, so that $\iota \circ \alpha^\lambda = \alpha^\lambda \circ \eta_\lambda$, for all λ, where ι is the identity mapping of Ω. Thus $\eta_\Omega = \iota$.

A subset Ω' of a unitary (R, S)-group Ω is a *free unitary basis* of Ω if for every unitary (R, S)-group Δ and for every mapping $f: \Omega' \to \Delta$ there exists one and only one R-homomorphism $\phi: \Omega \to \Delta$ whose restriction to Ω' is f. If the (R, S)-group Ω has such a basis, it is said to be *free unitary*. Note that such a group is not in general free. The free abstract groups are the free unitary $(Z, 1)$-groups.†

Using the methods leading to Theorem 5.1 one can prove the theorems:

THEOREM 5.7. *Ω is a free unitary (R, S)-group if and only if Ω is R-isomorphic to $e\overline{\Omega}$, where $\overline{\Omega}$ is a free (R, S)-group.*

THEOREM 5.8. *Let Ω' be a non-empty set of elements in the unitary (R, S)-group Ω; then Ω' is a free unitary basis if and only if*

(i) *for each ω in Ω' there exists an R-isomorphism $\{\omega\}_R \simeq R^+$ mapping ω onto e;*

(ii) *Ω is the free (R, S)-sum of the groups $\{\omega\}_R$, $\omega \in \Omega'$.*

Theorem 5.8 is an analogue of Theorem 5.1. One can also establish the analogue of Theorem 5.3 for unitary groups. Requiring in the diagram (5.2) all groups to be unitary we arrive at the notion of a *projective unitary* (R, S)-group, and now again Theorems 5.4, 5.5 can be reworded for the unitary case. Since η_Ω is always an idempotent R-endomorphism, we see that the projective unitary (R, S)-groups are the R-retracts under η of the projective (R, S)-groups, so that a unitary group is projective unitary if and only if it is projective.

† Recall that throughout we assumed all $(Z, 1)$-groups to be unitary.

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ON GROUPS OVER A D.G. NEAR-RING (II): CATEGORIES AND FUNCTORS

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IN the present paper the programme outlined in (**3**) is continued. Its principal aim is the development of a theory of categories and functors of R-groups and a more detailed treatment of some particular functors, which will subsequently be used for the introduction of satellite or derived functors of R-groups.

As in module theory the main interest lies in additive functors. The first step here is to define appropriately an addition of group homomorphisms when the underlying groups are no longer abelian. This can be done in essentially two ways, both arising out of a suitable generalization of homomorphism addition in module theory. The first of these leads to *natural addition* of group mappings [see, e.g., (**3**) § 1, and (**2**) 1, Example 3, where this was referred to as *addition of the first type*]. Always writing, as in (**3**), group operations additively, the sum of two mappings $f, g\colon \Omega \to \Delta$, written on the left, is given by

$$(f+g)\omega = f\omega + g\omega.$$

For non-abelian groups this sum of two homomorphisms need no longer be a homomorphism. Those group mappings $\Omega \to \Delta$ which are generated by addition and subtraction of R-homomorphisms will however form an additive group $\mathrm{HOM}_R(\Omega, \Delta)$. It is these groups which then provide the natural criterion for additivity of functors.

The formal theory of categories and functors as based on the groups $\mathrm{HOM}_R(\Omega, \Delta)$ is, however, not yet sufficiently general to cover the situations one has to expect. While additive covariant functors can be defined at this stage, we need a second type of addition, which may be called *free addition* for the introduction of additive contravariant functors. [In (**2**) 1, Example 5 this was referred to as *addition of the second type*.] The procedure leading to this operation is, however, no longer universal; a set $\mathfrak{H}$ of homomorphisms $\Omega \to \Delta$ has to satisfy special conditions to admit a free addition-structure. Writing, for reasons presently to be discussed, the homomorphisms on the right, Ω is required to possess a basis for $\mathfrak{H}$, i.e. a generating set Ω' such that

for f, f_1 in $\mathfrak{H}$ there exist elements $f_2, f' \in \mathfrak{H}$ with

$$\omega f' = -\omega f, \qquad \omega f_2 = \omega f + \omega f_1,$$

for all ω in Ω'. Defining $f+f_1 = f_2$ we then obtain the required group structure on $\mathfrak{H}$. An additive contravariant functor of a category with natural addition of mappings takes this addition into free addition on the homomorphisms of the image category.

Another important example of free addition, which accounts for the terminology, is provided by a free (R, S)-group Ω, free either in the absolute sense of [(**3**) § 5] or relative to a category. A free basis of Ω is then a basis for the R-homomorphisms $\Omega \to \Delta$, and this leads to a second type of homomorphism group to be denoted by $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Omega, \Delta)$.

The 'left-right' duality in ring theory and in module theory is lost when we go over to near-rings and R-groups, so that the question of writing mapping operators on the left or right acquires a deeper significance. We shall discuss this in some detail, considering a (left) category $\mathfrak{C}$ of R-groups, in which the mappings are written on the left to form the groups $\mathrm{HOM}_R(\Omega, \Delta)$. We obtain various types of additive groups: (i) the groups Ω of the category $\mathfrak{C}$, (ii) the additive group R^+ of R, (iii) the groups $\mathrm{HOM}_R(\Omega, \Delta)$. Between certain pairs A, B of such groups a multiplication is defined, and these multiplications are connected by associative laws. Moreover, whenever the products ab $(a \in A,\ b \in B)$ are defined, then we have the *universal right distributive law*

$$(a_1+a_2)b = a_1 b + a_2 b.$$

Left distributivity on the other hand breaks down. We retain, however, what we may call the *special left distributive law*:

the elements a with $a(b_1+b_2) = ab_1+ab_2$ generate the additive group A.

It is in fact this law which forms the basis for the extension of the concepts of module theory to the present situation and provides the necessary structure for a significant theory. We have already in (**2**) stressed the importance of this law of 'distributive generation' in the theory of near-rings.

In order to preserve the structural laws when considering a category with free addition—instead of natural addition—of homomorphisms, the mappings will now have to be written on the right. We are thus led to a second type of category, called a *right category*. Accordingly we get two kinds of covariant functors: 'left to left' and 'right to right' and two kinds of contravariant functors: 'left to right' and 'right to

left'. The ideas discussed here will be used in the next paper to introduce the concept of a right R-group.

The fundamental concepts will be developed in § 1 (left categories, the group HOM_R) and in §§ 3, 4 (right categories).

In § 2 the notion of a variety will be extended to (R, S)-groups. We shall introduce a certain class of functors, called *variety functors* and show that group varieties and variety functors stand in bi-unique correspondence. This approach may be of some interest even for abstract groups. Here it leads to a generalization of the basic results on varieties to R-groups.

The groups HOM_R are to be investigated in § 5, where it will be seen that they define an additive functor forming a natural generalization of the functor of module theory. In a later paper we shall still have to consider the generalization of the other basic functor in module theory, the tensor product.

1. Left categories and covariant functors

The notation is throughout the same as in (**3**). In particular R is a d.g. near-ring. Group-operations will be written additively, unless otherwise mentioned. The definition of an R-group is that in [(**3**) § 1 B].

The mappings of a group Ω into a group Δ—written on the left—form a group $\mathrm{MAP}(\Omega, \Delta)$, with the natural addition of mappings

$$(f+g)\omega = f\omega + g\omega, \quad \text{for all } \omega \text{ in } \Omega. \tag{1.1}$$

The product $g \circ f$ of mappings $f\colon \Omega \to \Delta$, $g\colon \Delta \to \Sigma$ is given by

$$(g \circ f)\omega = g(f\omega), \quad \text{for all } \omega \text{ in } \Omega. \tag{1.2}$$

This operation is associative. If A is a set of mappings $\Delta \to \Omega$ and B a set of mappings $\Sigma \to \Delta$, the symbol $A \circ B$ stands for the set of mappings $a \circ b$ $(a \in A,\ b \in B)$. The use of the symbol $a \circ b$, or $A \circ B$, is always to imply that the product is in fact defined. The identity mapping of a group Ω will be denoted by ι_Ω.

If Ω, Δ are R-groups, $\mathrm{hom}_R(\Omega, \Delta)$ is the set of R-homomorphisms $\Omega \to \Delta$, and $\mathrm{HOM}_R(\Omega, \Delta)$ is the subgroup of $\mathrm{MAP}(\Omega, \Delta)$ generated by this set. Unless Δ is abelian, the elements of $\mathrm{HOM}_R(\Omega, \Delta)$ are no longer necessarily homomorphisms. We have

$$\mathrm{hom}_R(\Delta, \Sigma) \circ \mathrm{hom}_R(\Omega, \Delta) \subseteq \mathrm{hom}_R(\Omega, \Sigma). \tag{1.3}$$

When $\qquad g_1, g_2 \in \mathrm{MAP}(\Delta, \Sigma), \qquad f \in \mathrm{MAP}(\Omega, \Delta),$

we get the 'universal' right distributive law

$$(g_1+g_2) \circ f = g_1 \circ f + g_2 \circ f, \tag{1.4}$$

while, when

$$g \in \hom_R(\Delta, \Sigma), \qquad f_1, f_2 \in \mathrm{MAP}(\Omega, \Delta),$$

we get the 'special' left distributive law

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2. \tag{1.5}$$

By (1.3)–(1.5) we now deduce

$$\mathrm{HOM}_R(\Delta, \Sigma) \circ \mathrm{HOM}_R(\Omega, \Delta) \subseteq \mathrm{HOM}_R(\Omega, \Sigma). \tag{1.6}$$

The mappings of $\mathrm{HOM}_R(\Omega, \Delta)$ will no longer commute with the operations of R. If however the element s of R induces an endomorphism on Δ, then, for all f in $\mathrm{HOM}_R(\Omega, \Delta)$ and all ω in Ω, $fs\omega = sf\omega$. This observation leads to the equation

$$\hom_Z(\Omega, \Delta) \cap \mathrm{HOM}_R(\Omega, \Delta) = \hom_R(\Omega, \Delta). \tag{1.7}$$

Here Z is the ring of integers, and $\hom_Z(\Omega, \Delta)$ is thus the set of homomorphisms of abstract groups.†

In fact Δ is an (R, S)-group for some distributive semigroup S in R generating R. It follows that, if $f \in \mathrm{HOM}_R(\Omega, \Delta)$, then f commutes with the operations of S. If in addition $f \in \hom_Z(\Omega, \Delta)$, this implies that f will commute with the operations of R.

A (left) *category*‡ $\mathfrak{C}$ of R-groups is a collection of R-groups containing the null group.§ An R-group Ω is *projective for* $\mathfrak{C}$ if every diagram

$$\begin{tikzcd} & \Omega \arrow[d] & \\ \Sigma \arrow[r] & \Delta \arrow[r] & 0 \end{tikzcd}$$

of R-homomorphisms with exact row and with Σ, Δ in $\mathfrak{C}$ can be completed to a commutative diagram

$$\begin{tikzcd} & \Omega \arrow[dl] \arrow[d] \\ \Sigma \arrow[r] & \Delta \end{tikzcd}$$

If in addition Ω lies in $\mathfrak{C}$, we say that Ω is $\mathfrak{C}$-projective. A *free basis for* $\mathfrak{C}$ in an R-group Ω is an R-generating set Ω' of Ω such that, for

† As in the preceding paper Z-groups are always understood to be unitary $(Z, 1)$-groups, i.e. abstract groups.

‡ We shall throughout adopt a naïve point of view and shall avoid going into the logical difficulties which arise in defining categories such as the category of R-groups. It may suffice that there are procedures to make such definitions rigorous. [Cf. (**1**).]

§ As it stands this definition is as yet incomplete. The R-groups are merely the objects of the category $\mathfrak{C}$ in the sense of Eilenberg–MacLane. See footnote ‖ on the next page.

every group Δ in $\mathfrak{C}$ and for every mapping $f: \Omega' \to \Delta$, there exists an R-homomorphism $\phi: \Omega \to \Delta$ extending the mapping f. If Ω has such a free basis, then Ω is *free for* $\mathfrak{C}$; if in addition $\Omega \in \mathfrak{C}$, then Ω is $\mathfrak{C}$-free. Clearly, if Ω is free for $\mathfrak{C}$, then Ω is projective for $\mathfrak{C}$.

If $\mathfrak{C}$ is the category of (R, S)-groups,† then a $\mathfrak{C}$-free group is the same as a free (R, S)-group in the sense of (**3**) § 5. There we defined a free basis Ω' of Ω as a subset of Ω such that for every group Δ in $\mathfrak{C}$ and every mapping $f: \Omega' \to \Delta$ there exists a unique R-homomorphism $\Omega \to \Delta$ extending f. But by [(**3**) 3.3, 5.1] Ω' is then an R-generating set, and hence is a free basis for $\mathfrak{C}$ in the sense of our definition here. Conversely, if Ω' is a free basis for $\mathfrak{C}$, then evidently the extension ϕ of a mapping $f: \Omega' \to \Delta$ is unique. A similar remark applies to the category of unitary (R, S)-groups.

A *functor* F of $\mathfrak{C}$ in some left category $\mathfrak{C}_1$ of R_1-groups is given by (*a*) a function associating with each group Ω in $\mathfrak{C}$ a group $F(\Omega)$ in $\mathfrak{C}_1$, (*b*) a function associating with each pair of groups Ω, Δ in $\mathfrak{C}$ a mapping‡

$$F: \hom_R(\Omega, \Delta) \to \hom_{R_1}(F(\Omega), F(\Delta)), \tag{1.8}$$

such that§

A (i) $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$,

(ii) $F(\iota_\Omega) = \iota_{F(\Omega)}$,

(iii) $F(\alpha)$ *is null if* α *is null.*

We shall also call F a *covariant functor* of $\mathfrak{C}$ without necessarily specifying the left image category $\mathfrak{C}_1$.

The functor F is said to be *additive* if the mappings (1.8) can be extended to homomorphisms‖

$$F: \mathrm{HOM}_R(\Omega, \Delta) \to \mathrm{HOM}_{R_1}(F(\Omega), F(\Delta)). \tag{1.9}$$

From (1.4), (1.5) it is easily deduced that the multiplicative rule A (i) will extend to mappings α, β in groups HOM_R.

† The symbol (R, S) is used as in (**3**). S is thus always a distributive semigroup in R generating R. The definition of an (R, S)-group is given in [(**3**) § 1].

‡ Without danger of confusion we may use the functor symbol also to denote the mappings (1.8).

§ A (iii) goes beyond the customary postulates for (non-additive) functors in module theory. It is convenient here to impose this additional postulate. There are important classes of non-additive functors of R-groups which all satisfy A (iii): such as the satellites of additive functors, to be introduced elsewhere.

‖ It is now seen that $\mathfrak{C}$ is properly defined in the sense of Eilenberg–MacLane. Strictly speaking we have two categories with the same objects. The abstract elements of one are the R-homomorphisms, the abstract elements of the other are the mappings in the groups HOM_R. No confusion can arise by not formally distinguishing between these two categories.

Let F be a functor of a left category $\mathfrak{C}$ in a left category $\mathfrak{C}_1$. If $\mathfrak{C}'$ is a subcategory of $\mathfrak{C}$ and $\mathfrak{C}_1$ a subcategory of $\mathfrak{C}_1^*$, then F defines in a natural manner a functor F' of $\mathfrak{C}'$ in $\mathfrak{C}_1^*$. We shall as a rule not distinguish between the two functors F and F'.

It will be useful to have the notions of a *subfunctor* and of a *quotient functor* of F [cf. (**1**)]. If G is a covariant functor of $\mathfrak{C}$, if the group $G(\Omega)$ is always a subgroup of $F(\Omega)$ and if $G(\alpha)$ is given by the restriction of $F(\alpha)$ to $G(\Omega)$, we call G a *subfunctor* of F. If moreover $G(\Omega)$ is always normal in $F(\Omega)$, then G is a *normal subfunctor* of F. In the latter case the groups $F(\Omega)/G(\Omega)$ define in the natural manner a quotient functor F/G. To define a subfunctor G of F it will suffice to specify subgroups $G(\Omega)$ of $F(\Omega)$, provided that it can be verified that, if $\alpha\colon \Omega \to \Delta$ is an R-homomorphism of groups in $\mathfrak{C}$, then $F(\alpha)$ maps $G(\Omega)$ into $G(\Delta)$. Let G_1, G_2 be normal subfunctors of F and put

$$(G_1+G_2)(\Omega) = G_1(\Omega)+G_2(\Omega),$$
$$(G_1 \cap G_2)(\Omega) = G_1(\Omega) \cap G_2(\Omega),$$
$$(G_1, G_2)(\Omega) = (G_1(\Omega), G_2(\Omega)).$$

Then

[1.1] *G_1+G_2, $G_1 \cap G_2$, (G_1, G_2) are normal subfunctors of F.*

We shall now derive some general properties of additive covariant functors. Almost immediately from the definition we get

[1.2] *The property of being additive is inherited by subfunctors and quotient functors. Preservation of monomorphisms (epimorphisms) is inherited by subfunctors (quotient functors).*

[1.2] will be of particular importance for the subfunctors and quotient functors of the identity functor I given by

$$I(\Omega) = \Omega, \qquad I(\alpha) = \alpha.$$

For any group Ω and any integer n, let $n\iota_\Omega$ be the n-multiple in $\mathrm{HOM}_Z(\Omega,\Omega)$ of ι_Ω, so that $(n\iota_\Omega)\omega = n\omega$ for ω in Ω. If F is an additive covariant functor, then

$$F(n\iota_\Omega) = n\iota_{F(\Omega)}.$$

If $n\iota_\Omega$ is an endomorphism of the R-group Ω, then by (1.7) it is an R-endomorphism. Hence $n\iota_{F(\Omega)}$ is an endomorphism of $F(\Omega)$.

Assume now that Ω is of finite exponent n. Then $n\iota_\Omega$ is null, i.e. $n\iota_{F(\Omega)}$ is null. Hence we have the theorem:

THEOREM 1.3. *If F is an additive covariant functor and Ω is of exponent n, then $F(\Omega)$ is of finite exponent dividing n.*

Next assume that Ω has abelian group structure. Then $-\iota_\Omega$ is an automorphism, and so $-\iota_{F(\Omega)}$ is an automorphism, whence $F(\Omega)$ is abelian. We have the theorem:

THEOREM 1.4. *An additive covariant functor F takes abelian values on abelian groups.*

Finally we prove the theorem:

THEOREM 1.5. *Let F be an additive covariant functor. If Ω is the direct sum of groups Ω_i ($i = 1,\ldots,n$) with injections α^i and projections π_i, then $F(\Omega)$ is the direct sum of the groups $F(\Omega_i)$ with injections $F(\alpha^i)$ and projections $F(\pi_i)$.*

For the proof we note that the mapping

$$\mathrm{HOM}_R(\Omega,\Omega) \to \mathrm{HOM}_{R_1}(F(\Omega), F(\Omega))$$

is now a homomorphism of near-rings. Theorem 1.5 will then be a consequence of the following criterion:

[1.6] *Let Ω, Ω_i ($i = 1,\ldots,n$) be groups in some left category $\mathfrak{C}'$, and let*

$$\alpha^i\colon \Omega_i \to \Omega, \qquad \pi_i\colon \Omega \to \Omega_i$$

be homomorphisms in $\mathfrak{C}'$. Let ι_k^i be the null mapping $\Omega_i \to \Omega_k$ if $i \neq k$, and the identity mapping $\Omega_i \to \Omega_i$ if $i = k$. Then Ω is the direct sum of the groups Ω_i with injections α^i and projections π_i if and only if

(i) $\pi_i \circ \alpha^k = \iota_i^k$ *for all i, k,*

(ii) $\alpha^1 \circ \pi_1 + \ldots + \alpha^n \circ \pi_n = \iota_\Omega$.

Proof. (i) is a necessary and sufficient criterion for Ω to be orthogonal sum [cf. (**3**) § 4]. If (ii) holds, then for ω in Ω we get

$$\omega = \alpha^1(\pi_1\omega) + \ldots + \alpha^n(\pi_n\omega).$$

Thus Ω is covered [cf. (**3**) § 4]; moreover we see from this representation, using (i), that Ω is filtered [cf. (**3**) § 4]. Thus Ω is a direct sum. The converse is immediate.

2. Varieties and variety functors

In this section $\mathfrak{C}$ is the left category of (R, S)-groups, or the left category of unitary (R, S)-groups. $\mathfrak{C}$ may more generally be taken as any left category of R-groups admitting the constructions and processes explicitly or implicity required for the definitions and theorems. Thus for the definition of varieties and variety functors, of covarieties and covariety functors and for the validity of Theorems 2.1–2.3 and Theorems 2.6, 2.7 it would suffice to take $\mathfrak{C}$ as the category of R-groups

or of unitary R-groups. All groups considered lie in $\mathfrak{C}$, and their homomorphisms are R-homomorphisms. The subcategories of $\mathfrak{C}$ to be considered are again left categories of R-groups.

A subcategory $\mathfrak{B}$ of $\mathfrak{C}$ is a *variety* if

B (i) *whenever* $\alpha: \Omega \to \Delta$ *is a homomorphism in* $\mathfrak{C}$ *and* $\Delta \in \mathfrak{B}$, *then* $\operatorname{Im} \alpha \in \mathfrak{B}$;

(ii) *whenever* $\alpha: \Omega \to \Delta$ *is a homomorphism in* $\mathfrak{C}$ *and* $\Omega \in \mathfrak{B}$, *then* $\operatorname{Im} \alpha \in \mathfrak{B}$;

(iii) *whenever* $\{\Omega_\lambda\}$ *is a family of groups in* $\mathfrak{B}$ *and* Ω *is their cartesian sum* [*cf.* (**3**) § 3], *then* $\Omega \in \mathfrak{B}$.

A covariant functor V of $\mathfrak{C}$ is a *variety functor* if

C (i) *V is a normal subfunctor of the identity functor I;*

(ii) *V preserves epimorphisms.*

By [1.2], V will be additive and will preserve monomorphisms. From the definition we have the theorem:

THEOREM 2.1. *If V_1, V_2 are variety functors, then so are V_1+V_2, (V_1, V_2), and $V_1 V_2$, provided that $V_1 V_2$ is normal in I.*

Here $V_1 V_2$ is the composite functor with groups

$$V_1 V_2(\Omega) = V_1(V_2(\Omega)).$$

With the usual definition of right-exactness we obtain the following characterization:

THEOREM 2.2. *A normal subfunctor V of I is a variety functor if and only if the quotient functor I/V is right-exact.*

We shall now establish a bi-unique correspondence between varieties and variety functors.

THEOREM 2.3. (i) *If V is a variety functor, then the groups Ω with $V(\Omega) = 0$ form a variety $\mathfrak{B}_V$.*

(ii) *If $\mathfrak{B}$ is a variety, then every group Ω has a minimal normal subgroup $V(\Omega)$ such that $\Omega/V(\Omega)$ lies in $\mathfrak{B}$. The groups $V(\Omega)$ determine a subfunctor $V_{\mathfrak{B}}$ of I, and this is a variety functor.*

(iii) $V = V_{\mathfrak{B}}$ *if and only if* $\mathfrak{B} = \mathfrak{B}_V$.

Proof. (i) Let $\mathfrak{B}$ be the collection of groups Ω with $V(\Omega) = 0$. By C (i), $V(0) = 0$, and so $\mathfrak{B}$ is a subcategory of $\mathfrak{C}$. By C (i) and [1.2], V preserves monomorphisms, whence $\mathfrak{B}$ satisfies B (i). By C (ii), $\mathfrak{B}$ will satisfy B (ii).

Let Ω be the cartesian sum of a family $\{\Omega_\lambda\}$ of groups in $\mathfrak{V}$ with defining projections π_λ. By hypothesis $V(\pi_\lambda)$ is null for all λ. But $V(\pi_\lambda)$ is given by the restriction of π_λ to $V(\Omega)$. Hence

$$V(\Omega) \subseteq \bigcap_\lambda \operatorname{Ker} \pi_\lambda = 0,$$

i.e. $\Omega \in \mathfrak{V}$.

(ii) Let $\mathfrak{V}$ be a variety. For every group Ω let $V(\Omega)$ be the intersection of the normal R-subgroups Δ_λ of Ω with Ω/Δ_λ in $\mathfrak{V}$. $V(\Omega)$ is a normal R-subgroup of Ω. There exists a natural R-monomorphism of $\Omega/V(\Omega)$ into the cartesian sum of the groups Ω/Δ_λ. By B (i)–(iii), $\Omega/V(\Omega) \in \mathfrak{V}$. Denote by π_Ω the projection $\Omega \to \Omega/V(\Omega)$.

Let $\alpha\colon \Omega \to \Delta$ be a homomorphism in $\mathfrak{C}$. By B (i), $\operatorname{Im}(\pi_\Delta \circ \alpha) \in \mathfrak{V}$, whence, by B (ii), $\operatorname{Coim}(\pi_\Delta \circ \alpha) \in \mathfrak{V}$, i.e. $\operatorname{Ker}(\pi_\Delta \circ \alpha) \supseteq V(\Omega)$. Hence α maps $V(\Omega)$ into $\operatorname{Ker} \pi_\Delta = V(\Delta)$. V is now seen to be a normal subfunctor of I.

Next let $\alpha\colon \Omega \to \Delta$ be an epimorphism. $\operatorname{Im} V(\alpha)$ is then normal in Δ, and α induces an epimorphism

$$\Omega/V(\Omega) \to \Delta/\operatorname{Im} V(\alpha).$$

By B (ii), $\Delta/\operatorname{Im} V(\alpha) \in \mathfrak{V}$, and so $V(\Delta) = \operatorname{Im} V(\alpha)$.

(iii) Let $\mathfrak{V}$ be a variety, $V = V_{\mathfrak{V}}$; then $\mathfrak{V}$ and $\mathfrak{V}_V$ both consist precisely of the groups Ω with $V(\Omega) = 0$.

Next let V be a variety functor; put $\mathfrak{V} = \mathfrak{V}_V$, $V_1 = V_{\mathfrak{V}}$. We consider the projections

$$\pi\colon \Omega \to \Omega/V(\Omega), \qquad \pi_1\colon \Omega \to \Omega/V_1(\Omega).$$

By C (i), $V(\pi)$ is null, whence, by C (ii), $\Omega/V(\Omega) \in \mathfrak{V}$, and so $V(\Omega) \supseteq V_1(\Omega)$. On the other hand by definition $V(\Omega/V_1(\Omega)) = 0$, and so $V(\pi_1)$ is null; therefore $V(\Omega) \subseteq V_1(\Omega)$.

Using the fact that every group Ω in $\mathfrak{C}$ has a representation by an exact sequence $\overline{\Omega} \to \Omega \to 0$ with $\overline{\Omega}$ $\mathfrak{C}$-free [cf. (**3**) 5.3] we get, for any variety $\mathfrak{V}$ with associated variety functor V, the theorem:

THEOREM 2.4. (i) *The $\mathfrak{V}$-free groups Ω are precisely those groups which have a representation by an exact sequence*

$$0 \to V(\overline{\Omega}) \to \overline{\Omega} \to \Omega \to 0,$$

where $\overline{\Omega}$ is $\mathfrak{C}$-free;

(ii) *the groups Ω in $\mathfrak{V}$ are precisely those groups which have a representation by an exact sequence*

$$\overline{\Omega} \to \Omega \to 0,$$

where $\overline{\Omega}$ is $\mathfrak{V}$-free;

(iii) *the $\mathfrak{V}$-projective groups are the R-retracts of the $\mathfrak{V}$-free groups.*

By Theorem 2.4, $\mathfrak{V}$ is completely determined by the groups $V(\Omega)$, where Ω lies in the category of $\mathfrak{C}$-free groups. We now show that the classical characterization of varieties in terms of fully invariant subgroups of free groups can be extended. The analogue of a fully invariant subgroup is that of a normal R-subgroup Σ of Ω which is mapped into itself by $\hom_R(\Omega, \Omega)$; such a subgroup will be called a *fully invariant* R-subgroup of Ω.

THEOREM 2.5. *If V is a variety functor and $\Omega \in \mathfrak{C}$, then $V(\Omega)$ is a fully invariant R-subgroup of Ω.*

A fully invariant R-subgroup Σ of a $\mathfrak{C}$-projective group Ω determines a variety $\mathfrak{V}$ by the law: $\Delta \in \mathfrak{V}$ if $\Sigma \subseteq \operatorname{Ker} \alpha$ for all α in $\hom_R(\Omega, \Delta)$. Also $\Sigma = V_{\mathfrak{V}}(\Omega)$.

Proof. The first assertion is immediate by the definition of a normal subfunctor of I.

Let then Σ and $\mathfrak{V}$ be given as in the second part of Theorem 2.5. $\mathfrak{V}$ is clearly a subcategory of $\mathfrak{C}$ satisfying B (i).

To establish B (ii) consider a homomorphism $\alpha: \Delta \to \Delta_1$ in $\mathfrak{C}$, with Δ in $\mathfrak{V}$. Ω being projective, any element $\beta \in \hom_R(\Omega, \operatorname{Im} \alpha)$ can be written as a product $\alpha' \circ \gamma$, where $\alpha': \Delta \to \operatorname{Im} \alpha$ is induced by α and

$$\gamma \in \hom_R(\Omega, \Delta).$$

Since $\Sigma \subseteq \operatorname{Ker} \gamma$, so also $\Sigma \subseteq \operatorname{Ker} \beta$.

For B (iii) let Δ be the cartesian sum of a family $\{\Delta_\lambda\}$ of groups in $\mathfrak{V}$ with defining projections π_λ. If $\alpha \in \hom_R(\Omega, \Delta)$, then, for all λ, $\pi_\lambda \circ \alpha \in \hom_R(\Omega, \Delta_\lambda)$ and so $\Sigma \subseteq \operatorname{Ker} \pi_\lambda \circ \alpha$. Hence

$$\alpha\Sigma \subseteq \bigcap_\lambda \operatorname{Ker} \pi_\lambda = 0.$$

Let $V = V_{\mathfrak{V}}$; denote by π the projection $\Omega \to \Omega/V(\Omega)$. Since $\Omega/V(\Omega) \in \mathfrak{V}$, we get

$$\Sigma \subseteq \operatorname{Ker} \pi = V(\Omega).$$

Next let π' be the projection $\Omega \to \Omega/\Sigma$. If $\alpha \in \hom_R(\Omega, \Omega/\Sigma)$, then, since Ω is projective, $\exists\, \beta \in \hom_R(\Omega, \Omega)$ with $\alpha = \pi' \circ \beta$. But

$$\beta\Sigma \subseteq \Sigma = \operatorname{Ker} \pi',$$

and so $\Sigma \subseteq \operatorname{Ker} \alpha$. Therefore $\Omega/\Sigma \in \mathfrak{V}$, i.e. $\Sigma \supseteq V(\Omega)$.

Examples. (1) The category of unitary (R, S)-groups is a variety of the category of (R, S)-groups.

(2) If $\mathfrak{a}$ is an ideal in R, the (unitary) (R, S)-groups annihilated by $\mathfrak{a}$ form a variety.

(3) If $\mathfrak{V}'$ is a variety of $(Z, 1)$-groups, i.e. a variety in the classical

sense, then the (R, S)-groups whose underlying abstract groups lie in $\mathfrak{V}^\times$ form a variety $\mathfrak{V}$ of (R, S)-groups.

There is a dual concept to that of a variety. A subcategory $\mathfrak{F}$ of $\mathfrak{C}$ is a *covariety* if it satisfies B (i), (ii) and

D. *Whenever* $\{\Omega_\lambda\}$ *is a family of groups in* $\mathfrak{F}$ *and* Ω *is their free* (R, S)-*sum* [cf. (**3**) § 3], *then* $\Omega \in \mathfrak{F}$.

A functor F of $\mathfrak{C}$ is a *covariety functor* if

E (i) F *is a subfunctor of* I.

(ii) *Whenever* $\alpha: \Delta \to \Omega$ *is a monomorphism in* $\mathfrak{C}$, *then*

$$\operatorname{Im} F(\alpha) = \operatorname{Im} \alpha \cap F(\Omega).$$

The covariety functors satisfy a dual law to Theorem 2.2.

THEOREM 2.6. *A covariety functor is left-exact.*

Dually to Theorem 2.3 one finds

THEOREM 2.7. (i) *If* F *is a covariety functor, the groups* Ω *with* $F(\Omega) = \Omega$ *form a covariety* $\mathfrak{F}^F$.

(ii) *If* $\mathfrak{F}$ *is a covariety, then every group* Ω *has a unique maximal subgroup* $F(\Omega)$ *lying in* $\mathfrak{F}$. *The groups* $F(\Omega)$ *define a subfunctor* $F^{\mathfrak{F}}$ *of* I, *and* $F^{\mathfrak{F}}$ *is a covariety functor.*

(iii) $F = F^{\mathfrak{F}}$ *if and only if* $\mathfrak{F} = \mathfrak{F}^F$.

Examples. (1) With the usual connotation of the symbol (R, S) let S be a multiplicative group. Let $F(\Omega)$ be the subgroup of Ω of fixed elements of S. F is a covariety functor. The corresponding covariety is given by the groups fixed under S.

(2) The unitary (R, S)-groups form a covariety of (R, S)-groups.

(3) If R is a ring with identity, $S = R^\times$ and $\mathfrak{a}$ an ideal in R, the unitary (R, S)-groups annihilated by $\mathfrak{a}$ form a covariety of (R, S)-groups.

3. Groups of right R-homomorphisms

In the present section we shall consider right R-homomorphisms. The mapping symbol is now to be written on the right; thus a right R-homomorphism $\Omega \to \Delta$ is characterized by the equations

$$\left.\begin{array}{ll} (\omega_1+\omega_2)f = \omega_1 f+\omega_2 f & (\omega_1, \omega_2 \in \Omega) \\ (x\omega)f = x(\omega f) & (\omega \in \Omega,\ x \in R) \end{array}\right\}. \tag{3.1}$$

The product $f \circ g$ is again defined by composition:

$$\omega(f \circ g) = (\omega f)g,$$

and is thus associative. With each left (right) R-homomorphism f there is associated its right (left) opposite f^r (f^l) by $\omega f^r = f\omega$ $(f^l\omega = \omega f)$. With the obvious meanings of the symbols involved we have the rules:

$$f^{rl} = f, \qquad f^{lr} = f;$$

$$(f \circ g)^r = g^r \circ f^r, \qquad (f \circ g)^l = g^l \circ f^l.$$

We denote the set of right R-homomorphisms $\Omega \to \Delta$ by $\hom_R^r(\Omega, \Delta)$.

So far our definition has been a purely formal one without real new conceptual content. We shall now, under certain hypotheses, introduce an addition of right R-homomorphisms which may be called *free addition* and which again will satisfy the universal right distributive law [cf. (3.1), [3.2]] and the special left distributive law [cf. (3.2), [3.3]]. A *group* $\mathfrak{F}$ *of right* R*-homomorphisms* $\Omega \to \Delta$ or briefly a group $\mathfrak{F}: \Omega \to \Delta$ is given by an additive group $\mathfrak{F}$ whose elements are right R-homomorphisms $\Omega \to \Delta$. An element $\omega \in \Omega$ is *distributive for* $\mathfrak{F}$ if

$$\omega(f_1+f_2) = \omega f_1+\omega f_2 \quad \text{for all } f_1, f_2 \text{ in } \mathfrak{F}. \tag{3.2}$$

We denote the set of these elements by $D(\Omega, \mathfrak{F})$ and postulate

F. *$D(\Omega, \mathfrak{F})$ is an R-generating set of Ω.*

This implies in particular that the zero of $\mathfrak{F}$ is the null mapping.

A *basis* Ω' in Ω for the group $\mathfrak{F}$ is a subset of $D(\Omega, \mathfrak{F})$ which is at the same time an R-generating set. To verify F it will of course suffice to find a basis in Ω for $\mathfrak{F}$.

The group Δ is an (R, S)-group for some S [cf. (3) § 1]. It is easily seen that

$$SD(\Omega, \mathfrak{F}) \subseteq D(\Omega, \mathfrak{F}). \tag{3.3}$$

By F it follows that $\quad SD(\Omega, \mathfrak{F}) \cup D(\Omega, \mathfrak{F})$

is a set of generators of the abstract group Ω. Hence, by (3.3),

[3.1] *If $\mathfrak{F}$ is a group of right R-homomorphisms $\Omega \to \Delta$, then $D(\Omega, \mathfrak{F})$ is a generating set of the abstract group Ω.*

The converse of [3.1] is trivial.

Let $\mathfrak{H}$ be a set of right R-homomorphisms $\Omega \to \Delta$. A *basis* Ω' for the set $\mathfrak{H}$ is an R-generating set of Ω such that (i) for $f \in \mathfrak{H}$ there is an $f' \in \mathfrak{H}$ with $\omega f' = -\omega f$ for all ω in Ω', (ii) for $f_1, f_2 \in \mathfrak{H}$ there is an $f \in \mathfrak{H}$ with $\omega f_1+\omega f_2 = \omega f$ for all $\omega \in \Omega'$. Then f is determined uniquely by f_1, f_2; the operation $f_1+f_2 = f$ will define a group structure, and the resulting group is a group $\mathfrak{H}/\Omega': \Omega \to \Delta$ with basis Ω'.

The following laws of composition are immediate:

[3.2] *Let* $\mathfrak{F}_i$ *be groups* $\Omega \to \Delta_i$ $(i = 1, 2)$ *both with basis* Ω'. *If*

$$g \in \hom_R^r(\Delta_1, \Delta_2) \text{ and } \mathfrak{F}_1 \circ g \subseteq \mathfrak{F}_2, \text{ then for all } f, h \in \mathfrak{F}_1,$$

$$(f+h) \circ g = f \circ g + h \circ g.$$

[3.3] *Let* $\mathfrak{F}_i$ *be groups* $\Delta_i \to \Omega$ *with bases* Δ_i' $(i = 1, 2)$. *If*

$$g \in \hom_R^r(\Delta_1, \Delta_2), \quad g \circ \mathfrak{F}_2 \subseteq \mathfrak{F}_1, \quad \Delta_1' g \subseteq \Delta_2',$$

then for all f, h *in* $\mathfrak{F}_2$

$$g \circ (f+h) = g \circ f + g \circ h.$$

If Δ has abelian group-structure, then $\mathrm{HOM}_R(\Omega, \Delta)$ consists precisely of the elements of $\hom_R(\Omega, \Delta)$. It is then easily seen that

[3.4] *If* Δ *is abelian and* $\mathfrak{F}$ *a group* $\Omega \to \Delta$, *then the mapping* $f \to f$ *induces a monomorphism*

$$\mathfrak{F} \to \mathrm{HOM}_R(\Omega, \Delta).$$

Assume that Ω has a basis Ω' for the set $\hom_R(\Omega, \Delta)$. We shall denote the resulting group $\Omega \to \Delta$ by $\mathfrak{HOM}_R(\Omega, \Delta)/\Omega'$. This group will still depend on the choice of Ω'. The following theorem leads however in many cases to essential uniqueness.

THEOREM 3.5. (i) *Let* Ω', Ω'' *be bases for* $\hom_R^r(\Omega, \Delta)$ *and let* $\Omega' \subseteq \Omega''$. *Then the identity mapping on* $\hom_R^r(\Omega, \Delta)$ *gives rise to an isomorphism*

$$\mathfrak{HOM}_R(\Omega, \Delta)/\Omega'' \simeq \mathfrak{HOM}_R(\Omega, \Delta)/\Omega'.$$

(ii) *Let* β *be a right R-automorphism of* Ω *and let* Ω' *be a basis for* $\hom_R^r(\Omega, \Delta)$. *Then* $\Omega'' = \Omega'\beta$ *is a basis for* $\hom_R^r(\Omega, \Delta)$, *and the mapping*

$$\beta_\Delta : f \to \beta \circ f$$

is an isomorphism

$$\mathfrak{HOM}_R(\Omega, \Delta)/\Omega'' \simeq \mathfrak{HOM}_R(\Omega, \Delta)/\Omega'.$$

Proof. (i) and the first part of (ii) are almost immediate from the definitions. For the last part denote addition in Δ by $+$, and addition in the two groups $\Omega \to \Delta$ by $+'$ and $+''$ respectively. For all ω in Ω' we have

$$\begin{aligned} \omega[\beta \circ (f+''g)] &= (\omega\beta)(f+''g) = (\omega\beta)f + (\omega\beta)g \\ &= \omega(\beta \circ f) + \omega(\beta \circ g) = \omega[\beta \circ f +' \beta \circ g]. \end{aligned}$$

Therefore $$\beta \circ f +' \beta \circ g = \beta \circ (f+''g),$$
as required.

THEOREM 3.6. *Let* Ω' *be a basis for* $\hom_R^r(\Omega, \Delta)$. *Then the mapping* $f \to f^r (f \in \hom_R(\Omega, \Delta))$ *sets up an epimorphism*

$$\rho : \mathrm{HOM}_R(\Omega, \Delta) \to \mathfrak{HOM}_R(\Omega, \Delta)/\Omega'.$$

Proof. We consider the subset H of elements $f \in \mathrm{HOM}_R(\Omega, \Delta)$ which have the property

$$\exists\, \rho f \in \mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Omega, \Delta)/\Omega'$$

with

$$f\omega = \omega[\rho f] \quad \text{for all } \omega \text{ in } \Omega'. \tag{3.4}$$

If ρf exists, it is of course uniquely determined by f. Clearly

$$\mathrm{hom}_R(\Omega, \Delta) \subseteq H;$$

when $f \in \mathrm{hom}_R(\Omega, \Delta)$, we have

$$\rho f = f^r. \tag{3.5}$$

If $f, g \in H$, then we verify easily that $f-g \in H$ and

$$\rho(f-g) = \rho f - \rho g. \tag{3.6}$$

Thus H is a subgroup of $\mathrm{HOM}_R(\Omega, \Delta)$. Since $\mathrm{hom}_R(\Omega, \Delta)$ is a generating set of the latter group, it follows that ρ is defined on all its elements and, by (3.6), (3.5), ρ is a homomorphism uniquely determined by the mapping $f \to f^r$. (3.5) then shows that ρ is an epimorphism.

If Δ is abelian, then the two types of addition of homomorphisms $\Omega \to \Delta$ coincide. More precisely we have, using Theorem 3.5 (i),

[3.7] *If Δ has abelian group structure, then Ω is a basis for* $\mathrm{hom}^r_R(\Omega, \Delta)$. *For any basis Ω' of Ω the group structure of $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Omega, \Delta)/\Omega'$ is uniquely determined by Ω and Δ. Thus ρ is an isomorphism.*

The groups $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Omega, \Delta)/\Omega'$ arise naturally if Ω is a $\mathfrak{C}$-free group with free basis Ω', $\mathfrak{C}$ being some left category, and if Δ is any group in $\mathfrak{C}$. Then Ω' is a basis for $\mathrm{hom}^r_R(\Omega, \Delta)$. If all free bases of Ω are of the same cardinal, then they are equivalent under right R-automorphisms. Hence by Theorem 3.5 (ii) the groups $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Omega, \Delta)/\Omega'$ are determined by Ω and Δ to within isomorphisms β_Δ. These isomorphisms β_Δ are moreover natural in the following sense. If $f \in \mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Omega, \Delta)/\Omega''$, $g \in \mathrm{hom}^r_R(\Delta, \Sigma)$ then

$$\beta_\Sigma(f \circ g) = (\beta_\Delta f) \circ g. \tag{3.7}$$

The preceding discussion applies also when Ω is a $\mathfrak{C}_1$-free group, Δ a group in a left category $\mathfrak{C}$, and $\mathfrak{C}_1$ is a covariety of $\mathfrak{C}$.

Examples. (1) Let $\Sigma = \Sigma(R, S)$ be the absolutely free (R, S)-group on one generator τ (cf. (**3**) § 5). Taking τ as a basis we obtain a group $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Sigma, \Delta)$ defined whenever Δ is an (R, S)-group. The mapping $f \to \tau f$ will then set up an isomorphism

$$\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(\Sigma, \Delta) \simeq \Delta. \tag{3.8}$$

(2) Let R be a d.g. near-ring with identity e and let $\mathfrak{C}$ be the category of R-groups on which e acts as an endomorphism. Let $\mathfrak{C}_1$ be the category of unitary R-groups. With e as a basis, R^+ is $\mathfrak{C}_1$-free and we

obtain groups $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(R^+, \Delta)$ when $\Delta \in \mathfrak{C}$. The mapping $f \to ef$ induces an isomorphism

$$\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(R^+, \Delta) \simeq e\Delta. \tag{3.9}$$

In particular, if $\Delta \in \mathfrak{C}_1$, then

$$\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(R^+, \Delta) \simeq \Delta. \tag{3.10}$$

(3) Let S be a multiplicative group, $R = Z(S)$ be the free near-ring on S [cf. (3) § 2]. In the category $\mathfrak{C}$ of S-groups the groups which are element-wise fixed under S form a covariety $\mathfrak{C}_1$. Considering Z as a group in $\mathfrak{C}_1$ we see that 1 is a free $\mathfrak{C}_1$-basis, giving rise to groups $\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(Z, \Delta)$ for Δ in $\mathfrak{C}$. If F is the associated covariety functor, then the mapping $f \to 1f$ induces an isomorphism

$$\mathfrak{H}\mathfrak{O}\mathfrak{M}_R(Z, \Delta) \simeq F(\Delta). \tag{3.11}$$

4. Right categories and functors

A *right category* $\mathfrak{C}$ of R-groups is given by (*a*) a collection $\mathfrak{C}$ of R-groups including the null group, (*b*) a function associating with each pair of groups Δ, Σ in $\mathfrak{C}$ a set $\mathfrak{C}(\Delta, \Sigma)$ of right R-homomorphisms $\Delta \to \Sigma$. The sets $\mathfrak{C}(\Delta, \Sigma)$ will be called the *mapping sets* of $\mathfrak{C}$. We postulate

G (i) $\mathfrak{C}(\Delta, \Sigma) \circ \mathfrak{C}(\Sigma, \Omega) \subseteq \mathfrak{C}(\Delta, \Omega)$,

(ii) $\iota_\Delta^r \in \mathfrak{C}(\Delta, \Delta)$,

(iii) $\mathfrak{C}(\Delta, \Sigma)$ *contains the null mapping.*

The category $\mathfrak{C}$ is *additive* if (i) the mapping sets are groups of right R-homomorphisms, now to be called the *mapping groups* of $\mathfrak{C}$, and if (ii) each Δ in $\mathfrak{C}$ has a subset Δ' which is a simultaneous basis for all the groups $\mathfrak{C}(\Delta, \Sigma)$. Δ' will be called a *basis of* Δ *for* $\mathfrak{C}$. It is clear that there will then exist a unique maximal basis of Δ for $\mathfrak{C}$, to be denoted by $D(\Delta)$. $D(\Delta)$ is the intersection of the sets $D(\Delta, \mathfrak{C}(\Delta, \Sigma))$ of distributive elements, for all Σ in $\mathfrak{C}$.

By [3.2] and G (i) an additive category $\mathfrak{C}$ satisfies the universal right distributive law, i.e., for Δ, Σ, Ω in $\mathfrak{C}$ and for all g_1, g_2 in $\mathfrak{C}(\Delta, \Sigma)$, all f in $\mathfrak{C}(\Sigma, \Omega)$,

$$(g_1+g_2) \circ f = g_1 \circ f + g_2 \circ f_1. \tag{4.1}$$

An element f in $\mathfrak{C}(\Delta, \Sigma)$ is *distributive* if, whenever $\Omega \in \mathfrak{C}$, $g_1, g_2 \in \mathfrak{C}(\Sigma, \Omega)$, then

$$f \circ (g_1+g_2) = f \circ g_1 + f \circ g_2. \tag{4.2}$$

The additive category $\mathfrak{C}$ is *distributively generated* if, for all Δ, $\Sigma \in \mathfrak{C}$, $\mathfrak{C}(\Delta, \Sigma)$ is generated by distributive elements.

From [3.3] we get for additive categories

[4.1] *The following three statements on an element f in $\mathfrak{C}(\Delta, \Sigma)$ are equivalent*:

(i) *f is distributive*;

(ii) *$\Delta' f \subseteq D(\Sigma)$ for some basis Δ' of Δ for $\mathfrak{C}$*;

(iii) *$D(\Delta) f \subseteq D(\Sigma)$.*

We can now complete the definition of a functor (cf. § 1). Let $\mathfrak{C}$ be a category of R-groups and $\mathfrak{C}_1$ a category of R_1-groups. A functor F of $\mathfrak{C}$ in $\mathfrak{C}_1$ is given by (*a*) a function associating with each group Ω in $\mathfrak{C}$ a group $F(\Omega)$ in $\mathfrak{C}_1$, (*b*) a function associating with each pair of groups Ω, Δ in $\mathfrak{C}$ a mapping $F\ (= F_{\Omega,\Delta})$ as follows:

$$\left.\begin{array}{ll} \text{(I)} & \mathfrak{C}\ \textit{left},\ \mathfrak{C}_1\ \textit{right} \\ & \qquad \hom_R(\Omega, \Delta) \to \mathfrak{C}_1(F(\Delta), F(\Omega)) \\ \text{(II)} & \mathfrak{C}\ \textit{right},\ \mathfrak{C}_1\ \textit{left} \\ & \qquad \mathfrak{C}(\Omega, \Delta) \to \mathrm{HOM}_{R_1}(F(\Delta), F(\Omega)) \\ \text{(III)} & \mathfrak{C},\ \mathfrak{C}_1\ \textit{right} \\ & \qquad \mathfrak{C}(\Omega, \Delta) \to \mathfrak{C}_1(F(\Omega), F(\Delta)) \end{array}\right\}. \tag{4.3}$$

The following postulates are to hold:

H (i) $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$;

(ii) $F(\iota_\Omega) = \iota_{F(\Omega)}$;

(iii) *$F(\alpha)$ is null if α is null.*

Here ι_Δ is to be written on the right or left as the case may be.

The case '$\mathfrak{C}$, $\mathfrak{C}_1$ left' not covered in (4.3) has already been dealt with in § 1. There exist thus two types of covariant functors, 'left to left' and 'right to right', and two types of contravariant functors, 'left to right' and 'right to left'.

For the purpose of defining additive functors left categories will be considered to be additive. Then let $\mathfrak{C}$, $\mathfrak{C}_1$ be additive categories. The functor F of $\mathfrak{C}$ in $\mathfrak{C}_1$ is said to be *additive* if the mapping F given by (4.3) gives rise to a homomorphism, replacing in case (I) $\hom_R$ by HOM_R. It can be verified that H (i) will still hold in this wider sense.

Subfunctors and quotient functors can again be defined as in § 1. It is, however, no longer true that a subfunctor of an additive functor is necessarily additive. On the other hand we have the theorem:

THEOREM 4.2. *A quotient functor of an additive functor is additive.*

Proof. For functors with values in a left category this is immediate. If F is an additive functor in a right category $\mathfrak{C}_1$, we can assume without loss of generality that all the groups in $\mathfrak{C}_1$ are values of F and that the mappings F of (4.3) are epimorphisms. If now G is a quotient functor of F, then it is a composite functor LF, where L is a quotient functor of the identity functor of $\mathfrak{C}_1$. It will then suffice to establish the additivity of L.

For each Δ in $\mathfrak{C}_1$, let Δ' be a basis in Δ for $\mathfrak{C}_1$. Denote by π_Δ the epimorphism $\Delta \to L(\Delta)$. If $\alpha \in \mathfrak{C}_1(\Delta, \Sigma)$, then $L(\alpha)$ is determined by

$$(\delta\pi_\Delta)L(\alpha) = \delta(\alpha \circ \pi_\Sigma),$$

where it suffices to consider only the elements δ of Δ'. A simple calculation now shows that $\Delta'\pi_\Delta$ is a basis for the homomorphisms $L(\alpha)$ $(\alpha \in \mathfrak{C}(\Delta, \Sigma))$. The additivity of L is immediate.

The definitions given here can in the usual manner be extended to functors of several variables. Thus a functor $F(\Omega_1, \Omega_2)$ in two variables will associate with each fixed Ω_1 a functor in the variable Ω_2, and vice versa, and moreover the mappings induced by homomorphisms $\Omega_1 \to \Omega_1'$ commute in the usual way with those induced by homomorphisms $\Omega_2 \to \Omega_2'$.

5. The functor HOM_R

Let Ω be a fixed R-group, $\mathfrak{C}$ a category of R-groups, and, for Δ in $\mathfrak{C}$, put

$$A(\Delta) = \mathrm{HOM}_R(\Omega, \Delta), \qquad a(\Delta) = \mathrm{hom}_R(\Omega, \Delta).$$

If $\alpha \in \mathrm{HOM}_R(\Delta, \Sigma)$, $f \in A(\Delta)$, then, by (1.6), $\alpha \circ f \in A(\Sigma)$. Define $A(\alpha)$ by

$$A(\alpha)f = \alpha \circ f.$$

Then, using (1.3)–(1.5) we get

[5.1] *A is an additive functor of $\mathfrak{C}$ in the left category of abstract groups.*

A preserves monomorphisms.

If the sequence $0 \to \Delta \xrightarrow{\alpha} \Delta' \xrightarrow{\beta} \Delta''$ is exact, then

$$\mathrm{Ker}\, A(\beta) \supseteq \mathrm{Im}\, A(\alpha) \supseteq \mathrm{Ker}\, A(\beta) \cap a(\Delta').$$

With the same connotation of $\mathfrak{C}$ and Ω, for Δ in $\mathfrak{C}$, put

$$B(\Delta) = \mathrm{HOM}_R(\Delta, \Omega), \qquad b(\Delta) = \mathrm{hom}_R(\Delta, \Omega).$$

If $\alpha \in \mathrm{HOM}_R(\Sigma, \Delta)$, $f \in B(\Delta)$, then $f \circ \alpha \in B(\Sigma)$. Put

$$fB(\alpha) = f \circ \alpha.$$

Denote the set of such mappings $B(\alpha)$ by $\mathfrak{B}(\Delta, \Sigma)$. Each element of this set is a right Z-homomorphism $B(\Delta) \to B(\Sigma)$, and, by (1.5), $b(\Delta)$

is a simultaneous basis for the sets $\mathfrak{B}(\Delta, \Sigma)$, $\Sigma \in \mathfrak{C}$; each such set can thus be made into a group $\Delta \to \Sigma$. Finally let $\mathfrak{b}(\Delta, \Sigma)$ be the subset of $\mathfrak{B}(\Delta, \Sigma)$ of elements $B(\alpha)$ with α in $\hom_R(\Sigma, \Delta)$. Then

[5.2] *The groups $B(\Delta)$ with mapping groups $\mathfrak{B}(\Delta, \Sigma)$ and simultaneous bases $b(\Delta)$ form an additive right category $\mathfrak{B}$ of abstract groups. The elements of $\mathfrak{b}(\Delta, \Sigma)$ are distributive and generate $\mathfrak{B}(\Delta, \Sigma)$; $\mathfrak{B}$ is thus distributively generated. B is an additive functor of $\mathfrak{C}$ in $\mathfrak{B}$.*

If the sequence $\Delta \xrightarrow{\alpha} \Delta' \xrightarrow{\beta} \Delta'' \to 0$ is exact, then

$$\operatorname{Ker} B(\alpha) \supseteq \operatorname{Im} B(\beta) \supseteq \operatorname{Ker} B(\alpha) \cap b(\Delta'),$$

$$\operatorname{Ker} B(\beta) = 0.$$

Finally, by the associativity of the mapping product we conclude that

[5.3] $\mathrm{HOM}_R(\Omega, \Delta)$ *is a functor in the variables Ω and Δ.*

Analogous results can be established also for $\mathfrak{HOM}_R(\Omega, \Delta)/\Omega'$. Thus, for fixed Ω and Ω', we obtain a covariant functor of the variable Δ in a right category $\mathfrak{C}$. The isomorphisms (3.8)–(3.11) then give rise to an equivalence of functors. The same is true for the isomorphisms derived in Theorem 3.5 from a change of basis.

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THEOREMS ON STRONG RIESZ SUMMABILITY

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1.1. LET $\sum_{n=1}^{\infty} a_n$ be a given infinite series, and λ_n a positive monotonic increasing function of n, tending to infinity with n. For $r > -1$, we write

$$A_\lambda^r(x) = \sum_{\lambda_n < x} (x-\lambda_n)^r a_n.$$

We also write $$A_\lambda^0(x) = A_\lambda(x).$$

When $k \geqslant 0$, the series $\sum a_n$ is said to be 'summable (R, λ, k) to the value s' (**2**), if

$$\lim_{x\to\infty} x^{-k} A_\lambda^k(x) = s.$$

The series $\sum a_n$ is said to be 'strongly summable (R, λ, k) with index q', or 'summable $[R, \lambda, k, q]$ to the sum s', where $k > 0$ and $q > 0$, if

$$\int^{\omega} |x^{-(k-1)} A_\lambda^{k-1}(x) - s|^q \, dx = o(\omega),\dagger$$

as $\omega \to \infty$ (**6**).‡ We write 'summability $[R, \lambda, k]$' for 'summability $[R, \lambda, k, 1]$'. Further, if $x^{-k} A_\lambda^k(x)$ is of bounded variation in (A, ∞), where A is a finite positive number and $k \geqslant 0$, then the series $\sum a_n$ is said to be 'absolutely summable (R, λ, k)', or 'summable $|R, \lambda, k|$' (**3**), (**4**).

1.2. For summability (R, λ, k) the following theorem was established by Hardy and Riesz.

THEOREM [(**2**) Theorem 20]. *If* $\lambda_1 > 0$ *and* $\sum a_n$ *is summable* (R, λ, k), *then* $\sum a_n \lambda_n^{-k}$ *is summable* (R, l, k), *where* $l_n = e^{\lambda_n}$.

The analogous theorem for summability $|R, \lambda, k|$ has been recently proved by Tatchell (**7**). The object of the present paper is to investigate the corresponding problem for strong summability. Theorem 1 of this paper gives that, if summability $[R, \lambda, k, q]$ $(q = 1)$ of the series $\sum a_n$ is given, then the summability factor $\{\lambda_n^{-k}\}$ makes the factored series $\sum a_n \lambda_n^{-k}$ summable $[R, l, k, q]$, as in the case of ordinary and absolute summabilities. But, when the index q is greater than unity, Theorem 1

† The lower limit of the integral may be any finite positive number. Here and elsewhere, in such cases, the lower limit is omitted.

‡ For another form of definition see (**1**) and (**5**). We remark that what is here described as 'summability $[R, \lambda, k, q]$' is the same as what Glatfield (**1**) calls 'summability $|R, \lambda, k-1|^q$'.

shows that even the factor $\{\lambda_n^{-k+1-1/q}\}$ serves the purpose. Moreover, Theorem 2 shows that a less restrictive condition than summability $[R, \lambda, k, q]$ $(q > 1)$ of $\sum a_n$ suffices for the summability $[R, l, k, q]$ of $\sum a_n \lambda_n^{-k}$.

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2.1. I prove the following theorems:

THEOREM 1. *If* $\lambda_1 > 0$, $q \geqslant 1$, $kq' > 1$, *and* $\sum a_n$ *is summable* $[R, \lambda, k, q]$, *then* $\sum a_n \lambda_n^{-k+1/q'}$ *is summable* $[R, l, k, q]$, *where* $l_n = e^{\lambda_n}$ *and* $q^{-1}+q'^{-1} = 1$.

THEOREM 2. *If* (i) $\sum a_n$ *is summable* $[R, \lambda, k]$ *to the sum zero and*

$$\text{(ii)} \quad \int^{\omega} |A_\lambda^{k-1}(x)|^q \, dx = o(\omega^{kq}),$$

as $\omega \to \infty$, *then* $\sum a_n \lambda_n^{-k}$ *is summable* $[R, l, k, q]$, *where* $l_n = e^{\lambda_n}$, $\lambda_1 > 0$, $q \geqslant 1$ *and* $kq' > 1$.

2.2. The following lemmas will be required in the proof of the theorems:

LEMMA 1 [(**2**) 27, Lemma 6]. *If* $k > 0$, $\mu < 1$, *and* $\mu \leqslant k$, *then*

$$A_\lambda^{k-\mu}(x) = \frac{\Gamma(k-\mu+1)}{\Gamma(k+1)\Gamma(1-\mu)} \int_0^x (x-t)^{-\mu} \frac{d}{dt} A_\lambda^k(t) \, dt.$$

LEMMA 2. *If* $$G(x) = \int_0^1 \phi(x, u) g(u) \, du.$$

then $$\int_c^\omega |dG(x)| \leqslant \limsup_{0<u<1} \int_c^\omega |d_x \phi(x, u)| \int_0^1 |g(u)| \, du.$$

This lemma can easily be seen to be a particular case of another, wherein we have ∞ in place of ω, given by Tatchell [(**7**) Lemma 1 (iii)].

2.3. *Proof of Theorem* 1. We suppose, without any loss of generality, that the sum of the series $\sum a_n$ is zero. We also observe that, since $kq' > 1$, the condition

$$\int^\omega |x^{-(k-1)} A_\lambda^{k-1}(x)|^q \, dx = o(\omega)$$

is equivalent to $$\int^\omega |A_\lambda^{k-1}(x)|^q \, dx = o(\omega^{(k-1)q+1}); \tag{2.1}$$

and we have to show that, for some value of s,

$$\int^{\omega} |D_l^{k-1}(x)-sx^{k-1}|^q\,dx = o\,(\omega^{(k-1)q+1}),$$

where $D_l^{k-1}(x)$ denotes the Riesz sum of type l for the series

$$\sum a_n \lambda_n^{-k+1/q'}.$$

Changing the variables, it is easy to see that the result to be obtained is

$$\int^{\omega} |D_l^{k-1}(e^x)-se^{(k-1)x}|^q e^x\,dx = o\,\{e^{\{(k-1)q+1\}\omega}\}.$$

From now onwards we drop the suffixes λ and l: that is to say, we write simply $A^r(x)$ and $D^r(x)$ instead of $A_\lambda^r(x)$ and $D_l^r(x)$. We have

$$\begin{aligned} D^{k-1}(e^x) &= \int_{l_1}^{e^x} (e^x-u)^{k-1}\,dD(u) \\ &= \int_{\lambda_1}^{x} (e^x-e^t)^{k-1}\,dD(e^t) \\ &= \int_{\lambda_1}^{x} (e^x-e^t)^{k-1}t^{-k+1/q'}\,dA(t). \end{aligned}$$

Case (i): *k an integer.*

Integrating by parts k times, we obtain

$$\begin{aligned} D^{k-1}(e^x) &= A^{k-1}(x)e^{(k-1)x}x^{-k+1/q'}+ \\ &\qquad +\frac{(-1)^k}{(k-1)!}\int_{\lambda_1}^{x} A^{k-1}(t)\left(\frac{\partial}{\partial t}\right)^k \{(e^x-e^t)^{k-1}t^{-k+1/q'}\}\,dt \\ &= I_1(x)+I_2(x), \quad \text{say.} \end{aligned}$$

It is easy to see that, by virtue of the hypothesis, i.e. condition (2.1),

$$\begin{aligned} \int^{\omega} |I_1(x)|^q e^x\,dx &= \int^{\omega} |A^{k-1}(x)e^{(k-1)x}x^{-k+1/q'}|^q e^x\,dx \\ &= O\{e^{(k-1)q\omega}\omega^{(-k+1/q')q}e^{\omega}\}o(\omega^{(k-1)q+1}) \\ &= o\,\{e^{\{(k-1)q+1\}\omega}\}. \end{aligned}$$

Hence, by Minkowski's inequality, we have only to show that, for some value of s,

$$I_2(x) = se^{(k-1)x}+J(x),$$

where

$$\int^{\omega} e^x|J(x)|^q\,dx = o\,\{e^{\{(k-1)q+1\}\omega}\}. \tag{2.2}$$

Now

$$I_2(x) = C\int_{\lambda_1}^{x} A^{k-1}(t)(e^x-e^t)^{k-1}t^{-2k+1/q'}\,dt + \\ + \sum C\int_{\lambda_1}^{x} A^{k-1}(t)(e^x-e^t)^{k-1-p}e^{pt}t^{-k-m+1/q'}\,dt,$$

where C denotes a constant which may be different at each occurrence and inside the sign of summation it may be different for each term of the sum. Further

$$I_2(x) = C\int_{\lambda_1}^{x} A^{k-1}(t)e^{(k-1)x}t^{-2k+1/q'}\,dt + \\ + \sum C\int_{\lambda_1}^{x} A^{k-1}(t)e^{(k-1-p)x}e^{pt}t^{-k-m+1/q'}\,dt \\ = se^{(k-1)x}+J(x),$$

where

$$s = C\int_{\lambda_1}^{\infty} A^{k-1}(t)t^{-2k+1/q'}\,dt \tag{2.3}$$

and

$$J(x) = C\int_{x}^{\infty} A^{k-1}(t)e^{(k-1)x}t^{-2k+1/q'}\,dt+\sum C\int_{\lambda_1}^{x} A^{k-1}(t)e^{(k-1-p)x}e^{pt}t^{-k-m+1/q'}\,dt \\ = CJ_1(x)+\sum CJ_{m,p}(x),$$

the summation being for $p = 1,\ldots, k-1$ and $m = 0,\ldots, k$. We now proceed to show that the integral on the right of (2.3) is convergent and that the condition (2.2) is satisfied.

Since [(**6**) Theorems 1, 4] summability $[R,\lambda,k,q]$ implies summability (R,λ,k) we have, under the hypothesis,

$$A^k(t) = o(t^k).$$

Hence $$C\int^{\infty} A^{k-1}(t)t^{-2k+1/q'}\,dt = C\int^{\infty} A^k(t)t^{-2k-1+1/q'}\,dt+C$$

is convergent. And, again by Minkowski's inequality, the condition (2.2) will be satisfied if

$$\int^{\omega} |J_1(x)|^q e^x\,dx = o\{e^{\{(k-1)q+1\}\omega}\}, \tag{2.4}$$

and

$$\int^{\omega} |J_{p,m}(x)|^q e^x\,dx = o\{e^{\{(k-1)q+1\}\omega}\}. \tag{2.5}$$

Now $$J_1(x) = o\,(e^{(k-1)x}),$$

whence (2.4) immediately follows; and, by virtue of the hypothesis and

use of Hölder's inequality, we obtain, for $q > 1$,

$$\int^{\omega} |J_{p,m}(x)|^q e^x\,dx$$

$$= \int^{\omega} e^{(k-1-p)qx} e^x\,dx \left\{ \int_{\lambda_1}^{x} A^{k-1}(t) e^{pt} t^{-k-m+1/q'}\,dt \right\}^q$$

$$= O\{e^{\{(k-1-p)q+1\}\omega}\} \int^{\omega} dx \int_{\lambda_1}^{x} |A^{k-1}(t)|^q\,dt \left\{ \int_{\lambda_1}^{x} e^{pq't} t^{(-k-m+1/q')q'}\,dt \right\}^{q-1}$$

$$= O\{e^{\{(k-1-p)q+1\}\omega}\}\, o\left\{ \int^{\omega} x^{(k-1)q+1} e^{pqx} x^{-(k-1)q-1}\,dx \right\}$$

$$= o\{e^{\{(k-1)q+1\}\omega}\}.$$

When $q = 1$, by virtue of the hypothesis, (2.5) can easily be seen to be true. This completes the proof of Theorem 1 for integral k.

Case (ii): *k not an integer and $k > 1$.*

We denote by $[k]$ the integral part of k. Integrating by parts $[k]$ times, we obtain

$$D^{k-1}(e^x) = \int_{\lambda_1}^{x} C\left(\frac{\partial}{\partial t}\right)^{[k]} \{(e^x - e^t)^{k-1} t^{-k+1/q'}\} A^{[k]-1}(t)\,dt$$

$$= C \int_{\lambda_1}^{x} A^{[k]-1}(t)(e^x - e^t)^{k-[k]-1} e^{[k]t} t^{-k+1/q'}\,dt +$$

$$+ \sum C \int_{\lambda_1}^{x} A^{[k]-1}(t)(e^x - e^t)^{k-1-p} e^{pt} t^{-k-m+1/q'}\,dt$$

$$= J_1(x) + J_2(x), \quad \text{say},$$

where the summation in $J_2(x)$ is for $p = 0$, $m = [k]$; $p = 1, 2, \ldots, [k]-1$, $m = 1, 2, \ldots, [k]-1$. Since $k > 1$, we have, by Lemma 1,

$$J_1(x) = C \int_{\lambda_1}^{x} (e^x - e^t)^{k-[k]-1} e^{[k]t} t^{-k+1/q'}\,dt \int_{\lambda_1}^{t} (t-u)^{[k]-k} \frac{d}{du} A^{k-1}(u)\,du$$

$$= C \int_{\lambda_1}^{x} \frac{d}{du} A^{k-1}(u)\,du \int_{s}^{x} (e^x - e^t)^{k-[k]-1} (t-u)^{[k]-k} e^{[k]t} t^{-k+1/q'}\,dt.$$

We put $t = x\tau + u(1-\tau)$; then

$$J_1(x) = C \int_{\lambda_1}^{x} \frac{d}{du} A^{k-1}(u)\,du \int_{0}^{1} \phi_1(x,u,\tau)\phi_2(x,u,\tau) g(\tau)\,d\tau,$$

where
$$\phi_1(x,u,\tau) = \left\{\frac{e^{(x-u)(1-\tau)}-1}{(x-u)(1-\tau)}\right\}^{k-[k]-1} = O(1),\dagger$$
$$\phi_2(x,u,\tau) = e^{(k-1)\{x\tau+u(1-\tau)\}}\{x\tau+u(1-\tau)\}^{-k+1/q'}$$
$$= O\{e^{(k-1)x}x^{-k+1/q'}\}$$
and
$$g(\tau) = (1-\tau)^{k-[k]-1}\tau^{[k]-k}.$$
Integration by parts gives
$$J_1(x) = CA^{k-1}(x)e^{(k-1)x}x^{-k+1/q'}+C\int\limits_{\lambda_1}^{x} A^{k-1}(u)\,du\int\limits_0^1 \frac{\partial}{\partial u}\{\phi_1\phi_2\}g(\tau)\,d\tau$$
$$= J_{1,1}(x)+J_{1,2}(x), \quad \text{say.}$$
Now, by virtue of the hypothesis,
$$\int\limits^{\omega} |J_{1,1}(x)|^q e^x\,dx = O\{e^{\{(k-1)q+1\}\omega}\,\omega^{-(k-1)q-1}\}\int\limits^{\omega} |A^{k-1}(x)|^q\,dx$$
$$= o\{e^{\{(k-1)q+1\}\omega}\}.$$
Also (by Hölder's inequality, when $q > 1$)
$$|J_{1,2}(x)|^q \leqslant C\int\limits_{\lambda_1}^{x} |A^{k-1}(u)|^q\,du\left|\int\limits_0^1 \frac{\partial}{\partial u}(\phi_1\phi_2)g(\tau)\,d\tau\right| \times$$
$$\times\left\{\int\limits_{\lambda_1}^{x} du\left|\int\limits_0^1 \frac{\partial}{\partial u}(\phi_1\phi_2)g(\tau)\,d\tau\right|\right\}^{q-1}.$$
Since
$$\int\limits_0^1 g(\tau)\,d\tau$$
is constant, we have, by virtue of Lemma 2,
$$\int\limits_{\lambda_1}^{x} du\left|\int\limits_0^1 \frac{\partial}{\partial u}(\phi_1\phi_2)g(\tau)\,d\tau\right| = \int\limits_{\lambda_1}^{x}\left|d_u\int\limits_0^1 \phi_1\phi_2 g(\tau)\,d\tau\right|$$
$$\leqslant C\limsup_{0<\tau<1}\int\limits_{\lambda_1}^{x}|d_u\phi_1\phi_2|.$$
When τ and x are fixed, $\phi_1(x,u,\tau)$ is a monotonic increasing function of u, and $\phi_2(x,u,\tau)$ is a monotonic function of u which is increasing from a certain fixed value of u onwards. Hence
$$C\limsup_{0<\tau<1}\int\limits_{\lambda_1}^{x}|d_u\phi_1\phi_2| = O\{\phi_1(x,x,\tau)\phi_2(x,x,\tau)\} = O\{e^{(k-1)x}x^{-k+1/q'}\}.$$

† The O's are throughout to be taken as applying uniformly in all variables for all values which occur except where some restriction in the range of validity is explicitly stated.

Also it is easily seen that $\dfrac{\partial\phi_1}{\partial u} = O(1)$

and $$\frac{\partial\phi_2}{\partial u} = O\{e^{(k-1)x}x^{-k+1/q'}\}.$$

Hence $|J_{1,2}(x)|^q = O\{e^{(k-1)qx}x^{-(k-1)q-1}\}\,o\,(x^{(k-1)q+1})$.

Therefore
$$\int^{\omega} |J_{1,2}(x)|^q e^x\,dx = o\int^{\omega} e^{\{(k-1)q+1\}x}\,dx = o\{e^{\{(k-1)q+1\}\omega}\}.$$

We now proceed to consider $J_2(x)$. Integrating once again by parts, we obtain
$$J_2(x) = \sum C \int_{\lambda_1}^{x} A^{[k]}(t)(e^x-e^t)^{k-1-p}e^{pt}t^{-k-m+1/q'}\,dt = \sum CJ_{p,m}(x),$$

where $p = 0$, $m = [k]+1$; $p = 1, 2,..., [k]$, $m = 1,..., [k]$. Since summability $[R,\lambda,k,q]$ implies summability (R,λ,k) [(**6**) Theorems 1, 4], under the hypothesis, by virtue of the first theorem of consistency for summability (R) (**2**), $A^{[k]+1}(t) = o\,(t^{[k]+1})$.

Therefore the integral
$$C\int_{\lambda_1}^{x} \frac{A^{[k]}(t)}{t^{k+[k]+1-1/q'}}\,dt = C\,\frac{A^{[k]+1}(x)}{x^{k+[k]+1-1/q'}} + C\int_{\lambda_1}^{x} \frac{A^{[k]+1}(t)}{t^{k+[k]+2-1/q'}}\,dt,$$

as $x \to \infty$, is convergent. If we denote the sum of this integral by s, we have
$$e^{-(k-1)x}\int^{x} (e^x-e^t)^{k-1}A^{[k]}(t)t^{-k-[k]-1+1/q'}\,dt \to s,$$
as $x \to \infty$. Therefore
$$J_{0,[k]+1}(x) = se^{(k-1)x}+o\,(e^{(k-1)x}).$$

It will now suffice to show that
$$\int^{\omega} |J_{p,m}(x)|^q e^x\,dx = o\{e^{\{(k-1)q+1\}\omega}\} \quad (p = 1,..., [k];\ m = 1,..., [k]).$$

We have (by Hölder's inequality, when $q > 1$)
$$|J_{p,m}(x)|^q = \int_{\lambda_1}^{x} |A^{[k]}(t)|^q(e^x-e^t)^{k-1-p}e^{pt}t^{(-k-m+1/q')q}\,dt \times$$
$$\times\left\{\int_{\lambda_1}^{x} (e^x-e^t)^{k-1-p}e^{pt}\,dt\right\}^{q-1}$$
$$= O\{e^{(k-1)(q-1)x}\}\int_{\lambda_1}^{x} |A^{[k]}(t)|^q(e^x-e^t)^{k-1-p}e^{pt}t^{(-k-m+1/q')q}\,dt.$$

Hence

$$\int^{\omega} |J_{p,m}(x)|^q e^x \, dx$$

$$= O\left\{ \int^{\omega} e^{(k-1)(q-1)x} e^x \, dx \int_{\lambda_1}^{x} |A^{[k]}(t)|^q (e^x - e^t)^{k-1-p} e^{pt} t^{(-k-m+1/q')q} \, dt \right\}$$

$$= O\{e^{(k-1)(q-1)\omega}\} \int_{\lambda_1}^{\omega} |A^{[k]}(t)|^q e^{pt} t^{(-k-m+1/q')q} \, dt \int_{t}^{\omega} (e^x - e^t)^{k-1-p} e^x \, dx$$

$$= O\{e^{(k-1)(q-1)\omega + (k-p)\omega} e^{p\omega} \omega^{-(k+m-1/q')q}\} \, o\,(\omega^{[k]q+1})$$

$$= o\{e^{\{(k-1)q+1\}\omega}\}.$$

This completes the proof of Case (ii) of Theorem 1.

Case (iii): $1/q' < k < 1$.

The required condition

$$\int^{\omega} |D^{k-1}(e^x) - s e^{(k-1)x}|^q e^x \, dx = o\{e^{\{(k-1)q+1\}\omega}\}$$

may now be rewritten as

$$\int^{\omega} \left| \frac{d}{dx} D^k(e^x) - s e^{kx} \right|^q dx = o\,(e^{kq\omega}).$$

Integrating by parts we obtain

$$D^k e^x$$

$$= \int_{\lambda_1}^{x} (e^x - e^t)^k t^{-k+1/q'} \, dA(t)$$

$$= \left(k - \frac{1}{q'}\right) \int_{\lambda_1}^{x} (e^x - e^t)^k t^{-k-1+1/q'} A(t) \, dt + k \int_{\lambda_1}^{x} (e^x - e^t)^{k-1} e^t t^{-k+1/q'} A(t) \, dt$$

$$= I_1(x) + I_2(x), \quad \text{say.} \tag{2.6}$$

Differentiating, we have

$$\frac{dI_1(x)}{dx} = C \int_{\lambda_1}^{x} A(t)(e^x - e^t)^{k-1} e^x t^{-k-1+1/q'} \, dt$$

$$= C \int_{\lambda_1}^{x} (e^x - e^t)^k t^{-k-1+1/q'} A(t) \, dt + C \int_{\lambda_1}^{x} (e^x - e^t)^{k-1} e^t t^{-k-1+1/q'} A(t) \, dt$$

$$= J_1(x) + J_2(x), \quad \text{say.} \tag{2.7}$$

Under the hypothesis, since summability $[R, \lambda, k, q]$ implies summability (R, λ, k) [(**6**) Theorems 1, 4], by the first theorem of consistency

for summability (R) (**2**), $$A^1(t) = o(t).$$
Therefore, since $kq' > 1$,
$$C\int^{\infty} A(t)t^{-k-1+1/q'}\,dt$$
is convergent. Hence, if we denote the sum of this integral by s, we have
$$J_1(x) = se^{kx}+o(e^{kx}). \tag{2.8}$$
Next, by Lemma 1,
$$J_2(x) = C\int_{\lambda_1}^{x} (e^x-e^t)^{k-1}e^t t^{-k-1+1/q'}\,dt \int_{\lambda_1}^{t} (t-u)^{-k}A^{k-1}(u)\,du$$
$$= C\int_{\lambda_1}^{x} A^{k-1}(u)\,du \int_{u}^{x} (e^x-e^t)^{k-1}(t-u)^{-k}e^t t^{-k-1+1/q'}\,dt.$$
We put $t = x\tau+u(1-\tau)$; then
$$J_2(x) = C\int_{\lambda_1}^{x} A^{k-1}(u)\,du \int_{0}^{1} \left(\frac{e^{x-t}-1}{x-t}\right)^{k-1} e^{kt}t^{-k-1+1/q'}(1-\tau)^{k-1}\tau^{-k}\,d\tau$$
$$= O\{e^{kx}x^{-k-1+1/q'}\}\int_{\lambda_1}^{x} |A^{k-1}(u)|\,du = o(e^{kx}), \tag{2.9}$$
since summability $[R,\lambda,k,q]$ $(q \geqslant 1)$ implies summability $[R,\lambda,k]$ [(**6**) Theorem 4]. Again, by Lemma 1, we have
$$I_2(x) = C\int_{\lambda_1}^{x} (e^x-e^t)^{k-1}e^t t^{-k+1/q'}\,dt \int_{\lambda_1}^{t} (t-u)^{-k}A^{k-1}(u)\,du$$
$$= C\int_{\lambda_1}^{x} A^{k-1}(u)\,du \int_{u}^{x} (e^x-e^t)^{k-1}(t-u)^{-k}e^t t^{-k+1/q'}\,dt.$$
Putting $t = x\tau+u(1-\tau)$, we obtain
$$I_2(x) = C\int_{\lambda_1}^{x} A^{k-1}(u)\,du \int_{0}^{1} \phi_1(x,u,\tau)\phi_2(x,u,\tau)g(\tau)\,d\tau,$$
where
$$\phi_1(x,u,\tau) = \left\{\frac{e^{(x-u)(1-\tau)}-1}{(x-u)(1-\tau)}\right\}^{k-1} = O(1),$$
$$\phi_2(x,u,\tau) = \frac{e^{k\{x\tau+u(1-\tau)\}}}{\{x\tau+u(1-\tau)\}^{k-1/q'}} = O\left(\frac{e^{kx}}{x^{k-1/q'}}\right),$$
and
$$g(\tau) = (1-\tau)^{k-1}\tau^{-k}.$$
We note that, since
$$\frac{e^y}{y-1}-\frac{1}{y}$$

is bounded for all positive y, we have

$$\frac{1}{\phi_1}\frac{\partial \phi_1}{\partial x} = (k-1)(1-\tau)\left\{\frac{e^{(x-u)(1-\tau)}}{e^{(x-u)(1-\tau)}-1} - \frac{1}{(x-u)(1-\tau)}\right\} = O(1). \tag{2.10}$$

Also
$$\frac{1}{\phi_2}\frac{\partial \phi_2}{\partial x} = k\tau + \frac{(-k+1/q')\tau}{x\tau+u(1-\tau)} = O(1). \tag{2.11}$$

Now, differentiating, we get

$$\frac{dI_2}{dx} = CA^{k-1}(x)e^{kx}x^{-k+1/q'} + C\int_{\lambda_1}^{x} A^{k-1}(u)\,du \int_0^1 \frac{\partial}{\partial x}\{\phi_1\phi_2\}g(\tau)\,d\tau$$
$$= CA^{k-1}(x)e^{kx}x^{-k+1/q'} + J_3(x), \quad \text{say.} \tag{2.12}$$

It is easy to see that, by virtue of the hypothesis,

$$C\int^{\omega} |A^{k-1}(x)e^{kx}x^{-k+1/q'}|^q\,dx = O\{e^{kq\omega}\omega^{-(k-1)q-1}\}\int^{\omega} |A^{k-1}(x)|^q\,dx$$
$$= o(e^{kq\omega}). \tag{2.13}$$

Also the estimates (2.10) and (2.11) show that

$$|J_3(x)| \leqslant C\int_{\lambda_1}^{x} |A^{k-1}(u)|K(x,u)\,du,$$

where
$$K(x,u) = \int_0^1 \phi_1\phi_2 g(\tau)\,d\tau.$$

When $q = 1$,
$$J_3(x) = O\{e^{kx}x^{-k}\}\int^{x} |A^{k-1}(u)|\,du = o(e^{kx}),$$

which together with (2.6), (2.7), (2.8), (2.9), (2.12), and (2.13) gives the required result. Now we have to consider the case when $q > 1$. We note that

$$\phi_1 = \begin{cases} O(1) & (x-t \leqslant 1), \\ O\{e^{(x-t)(k-1)}(x-t)^{1-k}\} & (x-t > 1), \end{cases}$$

since
$$\frac{e^y-1}{y} \to 1, \quad \text{as } y \to 0, \qquad \frac{e^y-1}{y} \sim \frac{e^y}{y}, \quad \text{as } y \to \infty.$$

Further,
$$\phi_2 \leqslant e^{kt}u^{-k+1/q'}.$$

If, now, we substitute these estimates in the integral for $J_3(x)$, we have to consider separately that part of the range of integration for which

$x-t \leqslant 1$, i.e. where $1-\tau \leqslant 1/(x-u)$, i.e. $\tau \geqslant 1-1/(x-u)$, and that part for which $x-t > 1$. We remark that, when $x-1 \leqslant u$, the first part includes the whole range of integration, so that the second part does not occur. We see that, for $x-1 \leqslant u \leqslant x$,

$$K(x,u) = O\left\{e^{kx}x^{-k+1/q'}\int_0^1 g(\tau)\,d\tau\right\} = O(e^{kx}x^{-k+1/q'}), \tag{2.14}$$

while, for $\lambda_1 \leqslant u \leqslant x-1$,

$$K(x,u) = O\left\{e^{kx}u^{-k+1/q'}\int_{1-1/(x-u)}^{1} g(\tau)\,d\tau\right\}+$$

$$+O\left\{\int_0^1 e^{(x-u)(1-\tau)(k-1)}\{(x-u)(1-\tau)\}^{1-k}e^{k\{x\tau+u(1-\tau)\}}u^{-k+1/q'}g(\tau)\,d\tau\right\}$$

$$= O\left\{e^{kx}u^{-k+1/q'}\int_{1-1/(x-u)}^{1} g(\tau)\,d\tau\right\}+$$

$$+O\left\{e^{(k-1)x+u}u^{-k+1/q'}(x-u)^{1-k}\int_0^1 e^{(x-u)\tau}\tau^{-k}\,d\tau\right\},$$

where

$$\int_{1-1/(x-u)}^{1} g(\tau)\,d\tau = \int_{1-1/(x-u)}^{1} (1-\tau)^{k-1}\tau^{-k}\,d\tau$$

$$= \int_{1-1/(x-u)}^{1} \left\{\frac{1}{\tau(1-\tau)}\right\}^{k+1}\tau(1-\tau)^{2k}\,d\tau$$

$$= \int_{1-1/(x-u)}^{1} \left\{\frac{1}{\tau}+\frac{1}{1-\tau}\right\}^{k+1}\tau(1-\tau)^{2k}\,d\tau$$

$$\leqslant C\left\{\int_{1-1/(x-u)}^{1} \tau^{-k}(1-\tau)^{2k}\,d\tau + \int_{1-1/(x-u)}^{1} \tau(1-\tau)^{k-1}\,d\tau\right\}$$

$$= O\{(x-u)^{-k}\},$$

and

$$(x-u)^{1-k}\int_0^1 e^{(x-u)\tau}\tau^{-k}\,d\tau = \int_0^{x-u} e^{v}v^{-k}\,dv = \int_0^1 e^{v}v^{-k}\,dv + \int_1^{x-u} e^{\frac{1}{2}v}v^{-k}e^{\frac{1}{2}v}\,dv$$

$$= O(1)+O\{e^{(x-u)}(x-u)^{-k}\}.$$

Hence, for $\lambda_1 \leqslant u \leqslant x-1$,

$$K(x,u) = O\{e^{kx}u^{-k+1/q'}(x-u)^{-k}\}. \tag{2.15}$$

We deduce from (2.14) and (2.15) that

$$\int_{\lambda_1}^{x} |K(x,u)|^{q'}\,du$$
$$= O\left\{e^{kq'x}\int_{\lambda_1}^{x-1} u^{-kq'+1}(x-u)^{-kq'}\,du\right\}+O\left\{e^{kq'x}x^{-kq'+1}\int_{x-1}^{x} du\right\}$$
$$= O\left\{e^{kq'x}\int_{\lambda_1}^{x-1}\left\{\frac{1}{x}\left(\frac{1}{u}+\frac{1}{x-u}\right)\right\}^{kq'} u\,du\right\}+O\{e^{kq'x}x^{-kq'+1}\}$$
$$= O\left\{e^{kq'x}x^{-kq'}\left(\int_{\lambda_1}^{x-1} u^{-kq'+1}\,du+\int_{\lambda_1}^{x-1}(x-u)^{-kq'}\,u\,du\right)\right\}+O\{e^{kq'x}x^{-kq'+1}\}$$
$$= O(e^{kq'x}x^{-kq'+1}).$$

Now an application of Hölder's inequality yields

$$|J_3(x)| \leqslant C\left\{\int_{\lambda_1}^{x} |A^{k-1}(u)|^{q}\,du\right\}^{1/q}\left\{\int_{\lambda_1}^{x} |K(x,u)|^{q'}\,du\right\}^{1/q'}$$
$$= o\,(x^{k-1+1/q}).\,O(e^{kx}x^{-k+1/q'}) = o\,(e^{kx}).$$

This, again, together with (2.6), (2.7), (2.8), (2.9), (2.12), and (2.13), gives the required result when $q > 1$. Thus the proof of Theorem 1 is complete.

2.4. *Proof of Theorem* 2. Inasmuch as the proof of this theorem follows by a slight modification of the proof given for Theorem 1, its details are omitted.

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